

ON KOTTWITZ' CONJECTURE FOR TWISTED INVOLUTIONS

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ABSTRACT. Kottwitz' conjecture is concerned with the intersections of Kazhdan–Lusztig cells with conjugacy classes of involutions in finite Coxeter groups. In joint work with Bonnafé, we have recently found a way to prove this conjecture for groups of type B_n and D_n . The argument for type D_n relies on two ingredients which were used there without proof: (1) a strengthened version of the “branching rule” and (2) the consideration of “ \diamond -twisted” involutions where \diamond is a graph automorphism. In this paper we deal with (1), (2) and complete the argument for type D_n ; moreover, we establish Kottwitz' conjecture for \diamond -twisted involutions in all cases where \diamond is non-trivial.

1. INTRODUCTION

Let W be a finite Coxeter group with generating S . We assume that we have a map $w \mapsto w^\diamond$ which is a group automorphism of W such that $S^\diamond = S$ and $(w^\diamond)^\diamond = w$ for all $w \in W$. An element $w \in W$ is called a \diamond -twisted involution if $w^\diamond = w^{-1}$. Given such an element $w \in W$, Kottwitz [11] defined a character Υ_w^\diamond of W which only depends on the \diamond -conjugacy class of w and which is remarkable for various reasons:

- (1) The decomposition of Υ_w^\diamond into irreducible characters is related to Lusztig's Fourier transforms [13, Chap. 4] associated with the various \diamond -stable “families” of $\text{Irr}(W)$.
- (2) By Lusztig and Vogan [21] there is a natural lift of Υ_w^\diamond to the generic one-parameter Iwahori-Hecke algebra associated with W, S . (By [19], there is even a version for arbitrary Coxeter groups.)
- (3) Kottwitz [11] conjectures that, for any left cell Γ of W in the sense of Kazhdan–Lusztig [10], the number of elements in the intersection of Γ with the \diamond -conjugacy class of w equals the scalar product of Υ_w^\diamond with the character afforded by Γ .

Following Kottwitz, we say that we are in the “split” case if $w^\diamond = w$ for all $w \in W$; otherwise, we are in the “quasi-split” case. If W is irreducible, then the quasi-split cases to consider are as follows:

- W of type A_n, E_6 and \diamond given by conjugation with the longest element in W ;
- W of type D_n and \diamond the non-trivial graph automorphism of order 2;
- W of type $F_4, I_2(m)$ and \diamond the non-trivial graph automorphism.

The results in this paper combined with previous work by Casselman [2], Kottwitz [11], Marberg [22], Bonnafé and the author [1], [4] will show that both the “split” and the “quasi-split” case of the conjecture in (3) hold for all W except possibly for type E_8 . (A. Halls at the University of Aberdeen is currently working on type E_8 .)

In Section 2, we introduce Kottwitz' involution module, both the split and the quasi-split version. In (2.6) we show that this coincides with the module constructed by Lusztig

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and Vogan [21]. (The identification in the split case already appeared in [6, §2].) We then also discuss various examples: first of all, the case where \diamond is given by conjugation with the longest element in W ; furthermore, the cases where W is of type F_4 , $I_2(m)$ and \diamond is the non-trivial graph automorphism.

In Section 3, which may be of independent interest, we clarify some notoriously troublesome issues concerning those irreducible characters of a Coxeter group of type D_n which are not invariant under the graph automorphism of order 2. The main result is Proposition 3.7 which establishes a strengthened version of “Pieri’s Rule” for these characters. This was used without proof in [1] to remove some ambiguities in the determination of the character of the split version of Kottwitz’ involution module for type D_n .

In Section 4, we consider Lusztig’s leading coefficients of character values of Iwahori–Hecke algebras. In [13, Chap. 12], [16], Lusztig has used his classification of the unipotent characters of a finite reductive group to determine the leading coefficients in the split case. Here, we extend at least some of these results to the quasi-split case. The main difficulty consists in carefully choosing extensions of \diamond -invariant characters of W to the semidirect product of W with $\langle \diamond \rangle \subseteq \text{Aut}(W)$. The applications to the quasi-split case in type D_n are contained in Proposition 4.11.

Finally, in Section 5, we complete the proof of Kottwitz’ conjecture for type D_n . The main idea is to treat the split and the quasi-split case at the same time. For this purpose, we develop a modified version of Kottwitz’ conjecture for type B_n , where we consider the left cells with respect to a suitable weight function in the sense of Lusztig [17]. The main result in this section is Theorem 5.3. This involves the construction of a modified version of Kottwitz’ involution module in Lemma 5.2. At first sight, this new combined setting seems to make things more complicated (which is certainly true from a technical point of view); but, in fact, I do not see any way how to carry out the argument separately for the split and the quasi-split case.

We shall use standard results and notation concerning the (complex) characters of finite groups. If χ, ψ are two class functions on a finite group G , then $\langle \chi, \psi \rangle_G$ denotes the usual scalar product for which the irreducible characters of G form an orthonormal basis. If $H \subseteq G$ is a subgroup and ψ is a class function on H , then $\text{Ind}_H^G(\psi)$ denotes the induced class function on G . We denote by $\text{Irr}(G)$ the set of all irreducible characters of G .

2. KOTTWITZ’ TWISTED INVOLUTION MODULE

Let W be a finite Coxeter group with generating set S . For $w \in W$, we denote by $\ell(w)$ the length of w . We assume that we have a map $w \mapsto w^\diamond$ which is an automorphism of W such that $S^\diamond = S$ and $(w^\diamond)^\diamond = w$ for all $w \in W$. We say that two elements $w, w' \in W$ are \diamond -conjugate if there exists some $x \in W$ such that $w' = x^\diamond w x^{-1}$. This defines an equivalence relation on W , and the corresponding equivalence classes will be called the \diamond -conjugacy classes of W . The subgroup

$$C_W^\diamond(w) := \{x \in W \mid x^\diamond w = wx\}$$

is called the \diamond -centraliser of w in W . Let Φ be the root system of W and $\Phi = \Phi^+ \amalg \Phi^-$ be the partition into positive and negative roots determined by S . Since $S^\diamond = S$, the automorphism $w \mapsto w^\diamond$ defines a permutation of the simple roots in Φ . We shall assume

that this permutation induces a map $\alpha \mapsto \alpha^\diamond$ on all of Φ such that

$$w^\diamond(\alpha^\diamond) = w(\alpha)^\diamond \quad \text{for all } w \in W \text{ and } \alpha \in \Phi.$$

Definition 2.1. An element $w \in W$ is called a “ \diamond -twisted involution” if $w^\diamond = w^{-1}$. Given such an element $w \in W$, let Φ_w be the set of all $\alpha \in \Phi$ such that $w(\alpha) = -\alpha^\diamond$. Then, by Kottwitz [11, 2.1, 4.2], we can define a linear character $\varepsilon_w: C_W^\diamond(w) \rightarrow \{\pm 1\}$ as follows. For $x \in C_W^\diamond(w)$ we have $\varepsilon_w(x) = (-1)^k$ where k is the number of positive roots $\alpha \in \Phi_w$ such that $x(\alpha)$ is negative. Then set

$$\Upsilon_w^\diamond := \text{Ind}_{C_W^\diamond(w)}^W(\varepsilon_w).$$

Remark 2.2. Let $C \subseteq W$ be any subset which is a union of \diamond -conjugacy classes of \diamond -twisted involutions in W . If C is a single \diamond -conjugacy class, then we certainly have $\Upsilon_w^\diamond = \Upsilon_{w'}^\diamond$ for all $w, w' \in C$. In general, we set

$$\Upsilon_C^\diamond = \sum_w \Upsilon_w^\diamond$$

where w runs over a set of representatives of the \diamond -conjugacy classes contained in C . In particular, this applies to the set of all \diamond -twisted involutions in W .

Remark 2.3. Assume that $w^\diamond = w$ for all $w \in W$; we just write this as $\diamond = 1$. Then a \diamond -twisted involution w is just an ordinary involution in W . Furthermore, Υ_w^\diamond is the character of the involution module in the “split case”; see [11, §2]. So, here, this character will be denoted by Υ_w^1 .

If \diamond is non-trivial, then Kottwitz [11] formulated Definition 2.1 in a slightly different way, using the semidirect product of W with the automorphism given by \diamond . The two versions are equivalent by the following remark.

Remark 2.4. Let \tilde{W} be the semidirect product of W with the subgroup $\langle \diamond \rangle \subseteq \text{Aut}(W)$. Thus, \tilde{W} is generated by W and an additional element γ such that $\gamma w \gamma = w^\diamond$ for all $w \in W$. (If $\diamond = 1$, then $\gamma = 1$ and $\tilde{W} = W$; otherwise, $\gamma^2 = 1$.) The natural action of W on Φ can be extended to \tilde{W} such that $\gamma(\alpha) = \alpha^\diamond$ for all $\alpha \in \Phi$.

First of all, this shows that $w \in W$ is a \diamond -twisted involution if and only if γw is an ordinary involution in \tilde{W} . Furthermore, two elements $w, w' \in W$ are \diamond -conjugate if and only if $\gamma w, \gamma w'$ are conjugate in \tilde{W} . Consequently, the map $w \mapsto \gamma w$ defines a bijection between the \diamond -conjugacy classes of W and the ordinary conjugacy classes of \tilde{W} which are contained in the coset $\gamma W \subseteq \tilde{W}$. We have $C_{\tilde{W}}(\gamma w) = C_W^\diamond(w)$ for all $w \in W$.

Remark 2.5. For any subset $I \subseteq S$ we denote by $W_I \subseteq W$ the corresponding parabolic subgroup and by w_I the longest element in W_I . Let C be a \diamond -conjugacy class of \diamond -twisted involutions. Then there exists a subset $I \subseteq S$ such that $w_I \in C$ and $s^\diamond w_I = w_I s$ for all $s \in I$; furthermore, $I^\diamond = I$ and w_I has minimum length in C . (See [7, Prop. 3.2.10] for the case where \diamond is the identity and He [8, Lemma 3.6] for the general case.) If we take $w = w_I$, then one easily sees that the set of roots Φ_{w_I} is just the parabolic subsystem $\Phi_I \subseteq \Phi$ corresponding to I . Let $\Pi_I \subseteq \Phi_I^+$ be the set of simple roots. Then we have:

(a) W_I is a normal subgroup of $C_W^\diamond(w_I)$.

Indeed, since $s^\diamond w_I = w_I s$ for all $s \in I$, we certainly have $W_I \subseteq C_W^\diamond(w_I)$. Now let $x \in C_W^\diamond(w_I)$ and $\alpha \in \Phi_I$. Then $w_I(x(\alpha)) = x^\diamond(w_I(\alpha)) = -x^\diamond(\alpha^\diamond) = -x(\alpha)^\diamond$ and so

$x(\alpha) \in \Phi_{w_I} = \Phi_I$, as required. Thus, (a) is proved. Consequently, by Howlett [9, Cor. 3], we have a semidirect product decomposition

$$(b) \quad C_W^\diamond(w_I) = Y \ltimes W_I \quad \text{where} \quad Y := \{y \in C_W^\diamond(w_I) \mid y(\Pi_I) = \Pi_I\}.$$

Note that Y is contained in the set of distinguished left coset representatives of W_I in W ; in particular, each element of Y sends all positive roots in Φ_I to positive roots. We conclude that

$$(c) \quad \varepsilon_{w_I}(yw) = (-1)^{\ell(w)} \quad \text{for all } y \in Y \text{ and } w \in W_I.$$

This provides an explicit description of $\Upsilon_{w_I}^\diamond$ which will be useful in several places below.

2.6. Let C be a \diamond -conjugacy class of \diamond -twisted involutions in W . Let M be a \mathbb{Q} -vector space with a basis $\{a_w \mid w \in C\}$. By Lusztig and Vogan [21, 7.1] (see also [19]), it is known that M is a $\mathbb{Q}[W]$ -module, where a generator $s \in S$ acts via the following formula:

$$s.a_w = \begin{cases} -a_w & \text{if } s^\diamond w = ws \text{ and } \ell(ws) < \ell(w), \\ a_{s^\diamond ws} & \text{otherwise.} \end{cases}$$

Let $w \in C$ be fixed. As discussed in [21, 6.3], we obtain a group homomorphism

$$\eta_w : C_W^\diamond(w) \rightarrow \{\pm 1\}$$

such that $x.a_w = \eta_w(x)a_w$ for all $x \in C_W^\diamond(w)$; furthermore, M is isomorphic to the $\mathbb{Q}[W]$ -module obtained by inducing η_w from $C_W^\diamond(w)$ to W . We claim that

$$\boxed{\Upsilon_w^\diamond \text{ is the character afforded by the } \mathbb{Q}[W]\text{-module } M.}$$

If \diamond is trivial, this is shown in [6, §2]. The argument in the general case is similar. Indeed, let $I \subseteq S$ and $w_I \in C$ be as in Remark 2.5. Then we have $C_W^\diamond(w_I) = Y \ltimes W_I$. Thus, it will be sufficient to show that

$$\eta_{w_I}(yw) = (-1)^{\ell(w)} \quad \text{for all } y \in Y \text{ and } w \in W_I.$$

We argue as in [6, Lemma 2.1]. For $s \in I$, we have $s.a_{w_I} = -a_{w_I}$ and so $\eta_{w_I}(s) = -1$. Consequently, we have $\eta_{w_I}(w) = (-1)^{\ell(w)}$ for all $w \in W_I$. Thus, it remains to show that $y.a_{w_I} = a_{w_I}$ for all $y \in Y$. We shall in fact show that $x.a_{w_I} = a_{x^\diamond w_I x^{-1}}$ where x is any distinguished left coset representative of W_I in W . We proceed by induction on $\ell(x)$. If $x = 1$, the assertion is clear. Now assume that $x \neq 1$ and choose $s \in S$ such that $\ell(sx) < \ell(x)$. By Deodhar's Lemma [7, 2.1.2], we also have that $z := sx$ is a distinguished left coset representative. Hence, using induction, we have $z.a_{w_I} = a_{z^\diamond w_I z^{-1}}$ and so

$$x.a_{w_I} = s.a_{z^\diamond w_I z^{-1}}.$$

Let $u = z^\diamond w_I z^{-1}$. Given the formula for the action of a generator on the basis elements of M , it now suffices to show that either $s^\diamond u \neq us$ or that $\ell(us) > \ell(u)$. Assume, if possible, that none of these two conditions is satisfied, that is, we have $s^\diamond u = us$ and $\ell(us) < \ell(u)$; in particular, $\ell(szw_I(z^\diamond)^{-1}) = \ell(su^{-1}) < \ell(u^{-1}) = \ell(zw_I(z^\diamond)^{-1})$. But then the ‘‘Exchange Lemma’’ (see [7, Exc. 1.6]) and the fact that $\ell(szw_I) = \ell(z) + \ell(w_I) + 1$ imply that $szw_I(z^\diamond)^{-1} = zw_I z'$ where $\ell(z') < \ell(z)$. Since $s^\diamond u = us$, we have $zw_I(z^\diamond)^{-1} s^\diamond = szw_I(z^\diamond)^{-1} = zw_I z'$ and so $(z^\diamond)^{-1} s^\diamond = z'$. This would imply that $\ell(z') = \ell((z^\diamond)^{-1} s^\diamond) = \ell((sz)^\diamond) = \ell(sz) > \ell(z)$, a contradiction. Hence, the assumption was wrong and so $x.a_{w_I} = a_{x^\diamond w_I x^{-1}}$, as required. Thus, the above claim is proved.

Next, we briefly recall the definition of Kazhdan–Lusztig cell modules.

2.7. Let \mathbf{H} be the generic one-parameter Iwahori–Hecke algebra associated with (W, S) , over the ring of Laurent polynomials $A = \mathbb{Z}[v, v^{-1}]$ in an indeterminate v . Thus, \mathbf{H} has a basis $\{T_w \mid w \in W\}$ and, for any $s \in S$ and $w \in W$, the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (v - v^{-1})T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $\{\mathbf{C}_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of \mathbf{H} . For any $w \in W$, we have

$$\mathbf{C}_w = T_w + \sum_{y \in W, y < w} (-1)^{\ell(w) + \ell(y)} v^{\ell(w) - \ell(y)} P_{y,w}(v^{-1}) T_y,$$

where $P_{y,w} \in \mathbb{Z}[v]$ are the polynomials defined in [10, Theorem 1.1] and $y < w$ denotes the Bruhat–Chevalley order. We write

$$\mathbf{C}_x \mathbf{C}_y = \sum_{z \in W} h_{x,y,z} \mathbf{C}_z \quad \text{where } h_{x,y,z} \in A \text{ for all } x, y, z \in W.$$

Let \leq_L be the pre-order relation on W defined in [10]; for any $w \in W$, we have

$$\mathbf{H} \mathbf{C}_w \subseteq \sum_{y \in W, y \leq_L w} A \mathbf{C}_y.$$

For $y, w \in W$, we write $y \sim_L w$ if $y \leq_L w$ and $w \leq_L y$. This defines an equivalence relation on W ; the equivalence classes are called the *left cells* of W . Let Γ be such a left cell. Let $[\Gamma]_A$ be a free A -module with a basis $\{e_x \mid x \in \Gamma\}$. By the definition of \sim_L , this is an \mathbf{H} -module where the action is given by

$$\mathbf{C}_x \cdot e_y = \sum_{z \in \Gamma} h_{x,y,z} e_z \quad \text{for all } x \in W \text{ and } y \in \Gamma.$$

Then we obtain a $\mathbb{Q}[W]$ -module $[\Gamma]_1$ by extension of scalars via the unique ring homomorphism $A \rightarrow \mathbb{Q}$ such that $v \mapsto 1$. We identify $[\Gamma]_1$ with its character.

Conjecture 2.8 (Kottwitz [11]). *Let $w \in W$ be a \diamond -twisted involution and C be its \diamond -conjugacy class in W . Let Γ be a left cell in W . Then*

$$\boxed{\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = |C \cap \Gamma|}.$$

Note that, by Remark 2.4, the above formulation is indeed equivalent to the original formulation by Kottwitz. The above formulation covers both the case where \diamond is the identity (“split” case) and the case where \diamond is non-trivial (“quasi-split” case).

Remark 2.9. The map \diamond induces an A -algebra automorphism $h \mapsto h^\diamond$ on \mathbf{H} such that $T_w^\diamond = T_{w^\diamond}$ for all $w \in W$. One easily checks that $\mathbf{C}_w^\diamond = \mathbf{C}_{w^\diamond}$ for all $w \in W$. Consequently, \diamond permutes the left cells of W , the right cells of W and the two-sided cells of W .

Now let $\Gamma \subseteq \mathfrak{C}$ be a left cell of W and C be a \diamond -conjugacy class of \diamond -twisted involutions in W . Then we claim that

$$C \cap \Gamma = \emptyset \quad \text{unless} \quad \mathfrak{C}^\diamond = \mathfrak{C}.$$

Indeed, assume that $\mathfrak{C}^\diamond \neq \mathfrak{C}$ and that there exists some $w \in C \cap \Gamma$. Then $w^{-1} = w^\diamond \in \Gamma^\diamond \subseteq \mathfrak{C}^\diamond$ and, hence, $w^{-1} \notin \mathfrak{C}$. But this contradicts [13, Lemma 5.2(iii)] which shows that, for any $w \in \mathfrak{C}$, we also have $w^{-1} \in \mathfrak{C}$. Thus, the above claim is proved.

This statement provides a first test case for Conjecture 2.8: we will have to show that then we also have $\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = 0$ for $w \in C$; see Remark 3.13 below for type D_n .

Example 2.10. Let $w_0 \in W$ be the longest element and assume that w_0 is not central in W . Then define $w^\diamond = w_0 w w_0$ for $w \in W$. One easily checks that the following hold.

- (a) $w \in W$ is a \diamond -twisted involution if and only if $w_0 w$ is an involution.
- (b) Let $w \in W$ be a \diamond -twisted involution and C its \diamond -conjugacy class. Then $C_W^\diamond(w) = C_W(w_0 w)$ and $C = w_0 \mathcal{C}$ where \mathcal{C} is the ordinary conjugacy class of $w_0 w$ in W .

Now let $w \in W$ be a \diamond -twisted involution and Γ be a left cell in W . Then Γw_0 and $w_0 \Gamma$ are also left cells and we have $[w_0 \Gamma]_1 = [\Gamma w_0]_1 = [\Gamma]_1 \otimes \varepsilon$; see [13, 5.14], [17, 11.7]. Consequently, we obtain

$$|C \cap \Gamma| = |w_0(\mathcal{C} \cap w_0 \Gamma)| = |\mathcal{C} \cap w_0 \Gamma|,$$

where C is the \diamond -conjugacy class of w and \mathcal{C} is the ordinary conjugacy class of $w_0 w$ in W . On the other hand, by [11, 5.3.1], we have

$$\langle \Upsilon_{w_0 w}^1, \chi \otimes \varepsilon \rangle_W = \langle \Upsilon_w^\diamond, \chi \rangle_W \quad \text{for all } \chi \in \text{Irr}(W).$$

This yields that

$$\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = \langle \Upsilon_{w_0 w}^1, [\Gamma]_1 \otimes \varepsilon \rangle_W = \langle \Upsilon_{w_0 w}^1, [w_0 \Gamma]_1 \rangle_W.$$

So, if the split version of Kottwitz' conjecture holds for W , then we have

$$\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = \langle \Upsilon_{w_0 w}^1, [w_0 \Gamma]_1 \rangle_W = |C \cap w_0 \Gamma| = |C \cap \Gamma|,$$

that is, the quasi-split version (with respect to \diamond) also holds.

This discussion applies, in particular, to (W, S) of type A_n , D_{2n+1} , E_6 , $I_2(2m+1)$. The split version of Kottwitz' conjecture holds in type A_n , as already observed by Kottwitz himself [11]; see also [1, Exp. 4.10]. For type D_{2n+1} , see [1, Cor. 7.6] and Corollary 5.4 below. Finally, Casselman [2] has verified that the split version holds in type E_6 ; see Marberg [22] for the dihedral groups.

Example 2.11. Let $m \geq 2$ and (W, S) be of type $I_2(2m)$ where $S = \{s_1, s_2\}$ and $s_1 s_2$ has order $2m$. Assume that $s_1^\diamond = s_2$ and $s_2^\diamond = s_1$. By Remark 2.5, it is clear that, up to \diamond -conjugacy, 1 is the only \diamond -twisted involution; let C_1 denote its \diamond -conjugacy class. Let us consider the corresponding character Υ_1^\diamond . We have

$$\text{Irr}(W) = \{1, \varepsilon, \varepsilon_1, \varepsilon_2, \chi_1, \chi_2, \dots, \chi_{m-1}\}$$

where 1 is the trivial character, ε is the sign character, ε_1 and ε_2 are two further characters of degree 1 and each χ_j has degree 2; see [7, 5.3.4]. Here, the notation is such that $\varepsilon_1(s_1) = \varepsilon_2(s_2) = 1$ and $\varepsilon_1(s_2) = \varepsilon_2(s_1) = -1$; furthermore, χ_j is determined by the condition that $\chi_j(1_2) = 2 \cos(\pi j/m)$. Now note that

$$C_W^\diamond(1) = \{x \in W \mid x^\diamond = x\} = \{1, 1_{2m}\}.$$

Furthermore, $\Phi_1 = \emptyset$ and so ε_1 is the trivial character of $C_W^\diamond(1)$. Then we find that

$$\Upsilon_1^\diamond = \text{Ind}_{C_W^\diamond(1)}^W(1) = \begin{cases} 1 + \varepsilon + \varepsilon_1 + \varepsilon_2 + \sum_{1 \leq j \leq (m-2)/2} 2\chi_{2j} & \text{if } m \text{ is even,} \\ 1 + \varepsilon + \sum_{1 \leq j \leq (m-1)/2} 2\chi_{2j} & \text{if } m \text{ is odd.} \end{cases}$$

Next, we consider the left cells of W . To simplify notation, write $1_k = s_1 s_2 s_1 \cdots$ (k factors) and $2_k = s_2 s_1 s_2 \cdots$ (k factors); in particular, $1_{2m} = 2_{2m}$ is the longest element in W . By [17, 8.8], the left cells are

$$\begin{aligned}\Gamma_0 &:= \{1_0\}, & \Gamma_1 &:= \{1_1, 2_2, 1_3, \dots, 1_{2m-1}\}, \\ \Gamma_2 &:= \{2_1, 1_2, 2_3, \dots, 2_{2m-1}\}, & \Gamma_{2m} &:= \{1_{2m}\}.\end{aligned}$$

Thus, we have

$$|C_1 \cap \Gamma_i| = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 2m, \\ m-1 & \text{if } i = 1 \text{ or } i = 2. \end{cases}$$

The characters of the left cell modules are given by:

$$[\Gamma_0]_1 = 1, \quad [\Gamma_1]_1 = \varepsilon_1 + \sum_{1 \leq j \leq m-1} \chi_j, \quad [\Gamma_2]_1 = \varepsilon_2 + \sum_{1 \leq j \leq m-1} \chi_j, \quad [\Gamma_{2m}]_1 = \varepsilon;$$

see, for example, [5, Exp. 2.2.8]. Consequently, we have

$$\langle \Upsilon_1^\diamond, [\Gamma_i]_1 \rangle_W = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 2m, \\ m-1 & \text{if } i = 1 \text{ or } i = 2. \end{cases}$$

Hence, we see that Kottwitz' conjecture holds in this case.

Example 2.12. Let (W, S) be of type F_4 , where $S = \{s_0, s_1, s_2, s_3\}$ is such that $s_0 s_1$ and $s_2 s_3$ have order 3 and $s_1 s_2$ has order 4. Assume that $s_0^\diamond = s_3$, $s_1^\diamond = s_2$, $s_2^\diamond = s_1$ and $s_3^\diamond = s_0$. Let $C_1 = \{x^\diamond x^{-1} \mid x \in W\}$ be the \diamond -conjugacy class containing $w = 1$. Using Remark 2.5 we find that C_1 is the only \diamond -conjugacy class of \diamond -twisted involutions in W . We have $C_W^\diamond(1) = \{x \in W \mid x^\diamond = x\}$; this is a dihedral group of order 16, generated by $s_0 s_3$ and $(s_1 s_2)^2$. Since $\Phi_1 = \emptyset$, we obtain by a direct computation (which can be done by hand):

$$\Upsilon_1^\diamond = \text{Ind}_{C_W^\diamond(1)}^W(1) = 1_1 + 1_4 + 2_1 + 2_2 + 2_3 + 2_4 + 2 \cdot 4_1 + 9_1 + 9_2 + 9_3 + 9_4 + 6_1 + 12_1,$$

where we use the notation for $\text{Irr}(W)$ as in [7, Table C.3 (p. 413)]. Using a computer algebra system capable of computing Kazhdan–Lusztig cells, it is straightforward to check that Conjecture 2.8 holds. For example, using PyCox [4], the left cells of W and the characters of the corresponding left cell modules are obtained by the following commands:

```
>>> W=coxeter("F",4); l=klcells(W,1,v)[0]
>>> ch=[leftcellleadingcoeffs(W,1,v,c)]['char'] for c in l]
>>> chartable(W) ['charnames']
```

The last command gives the labelling of $\text{Irr}(W)$. The set C_1 is obtained by:

```
>>> p=[3,2,1,0] # the permutation on S={0,1,2,3}
>>> C1=noduplicates([W.reducedword(w[:-1])+[p[s] for s in w],W)
                    for w in allwords(W)]
```

Further explanations are available through the online help in PyCox.

Remark 2.13. The map $w \mapsto w^\diamond$ on W also induces an operation on $\text{Irr}(W)$, which we denote by $\chi \mapsto \chi^\diamond$. We have $\chi^\diamond(w) = \chi(w^\diamond)$ for all $w \in W$. Let $\tilde{W} = \langle W, \gamma \rangle$ be the semidirect product as in Remark 2.4. By standard results on Clifford theory, we have

$$\text{Ind}_{\tilde{W}}^{\tilde{W}}(\chi) \in \text{Irr}(\tilde{W}) \quad \text{for all } \chi \in \text{Irr}(W) \text{ such that } \chi^\diamond \neq \chi.$$

On the other hand, let us denote

$$\mathrm{Irr}^\diamond(W) := \{\chi \in \mathrm{Irr}(W) \mid \chi^\diamond = \chi\}.$$

Then each $\chi \in \mathrm{Irr}^\diamond(W)$ has exactly two extensions to \tilde{W} , which differ only by a sign on elements in the coset γW .

Remark 2.14. Let \mathfrak{C} be a two-sided cell of W . We set

$$\mathrm{Irr}(W \mid \mathfrak{C}) = \{\chi \in \mathrm{Irr}(W) \mid \langle [\Gamma]_1, \chi \rangle_W \neq 0 \text{ for some left cell } \Gamma \subseteq \mathfrak{C}\}.$$

Alternatively, we have $\mathrm{Irr}(W \mid \mathfrak{C}) := \{\chi \in \mathrm{Irr}(W) \mid \chi \sim_{LR} w \text{ for some } w \in \mathfrak{C}\}$, where the relation “ $\chi \sim_{LR} w$ ” is defined in [13, 5.1 (p. 139)]. (This easily follows from the definitions; see, for example, [13, 2.2.18].) Thus, we obtain a partition

$$\mathrm{Irr}(W) = \coprod_{\mathfrak{C}} \mathrm{Irr}(W \mid \mathfrak{C}).$$

where \mathfrak{C} runs over all two-sided cells in W . Now recall from Remark 2.9 that \diamond permutes the left cells of W . One easily sees that, for a left cell Γ of W , we have

$$\mathrm{trace}(T_w, [\Gamma^\diamond]_A) = \mathrm{trace}(T_{w^\diamond}, [\Gamma]_A) \quad \text{for all } w \in W.$$

This certainly implies that $\mathrm{Irr}(W \mid \mathfrak{C}^\diamond) = \{\chi^\diamond \mid \chi \in \mathrm{Irr}(W \mid \mathfrak{C})\}$.

Remark 2.15. Having dealt with type F_4 and the dihedral groups, we shall assume from now on that W is a Weyl group and that \diamond is “ordinary” in the sense of [13, 3.1], that is, whenever $s, t \in S$ are in the same \diamond -orbit, then the product st has order 2 or 3. This has the following consequences.

- (a) Each $\chi \in \mathrm{Irr}(W)$ can be realised over \mathbb{Q} . (This is a well-known fact; see, for example, [7, 6.3.8].)
- (b) The two extensions of any $\chi \in \mathrm{Irr}^\diamond(W)$ to \tilde{W} can also be realised over \mathbb{Q} . (See [13, Prop. 3.2].)
- (c) If \mathfrak{C} is a two-sided cell of W such that $\mathfrak{C}^\diamond = \mathfrak{C}$, then $\mathrm{Irr}(W \mid \mathfrak{C}) \subseteq \mathrm{Irr}^\diamond(W)$. (See [13, 4.17].)

In any case, as far as the quasi-split version of Kottwitz’ conjecture is concerned, it now remains to deal with type D_n and the non-trivial graph automorphism of order 2.

3. ON THE IRREDUCIBLE CHARACTERS IN TYPE D_n

In this section we fix some notation concerning the irreducible characters of Coxeter groups of classical type. This is especially relevant in type D_n for n even, where there are characters which are not invariant under the graph automorphism of order 2; it will be important for us to know exactly how to distinguish these characters from each other. In Corollary 3.8, we establish a strengthened “branching rule” for type D_n . Then, in (3.9) and Proposition 3.12, we state explicit formulae for the decomposition of Υ_w^1 and Υ_w^\diamond .

3.1. For $\chi \in \mathrm{Irr}(W)$, let \mathbf{b}_χ denote the smallest integer $i \geq 0$ such that χ occurs in the i th symmetric power of the standard reflection representation of W . For example, if ε is the sign character of W , then

$$\mathbf{b}_\varepsilon = |T| = \ell(w_0)$$

where $w_0 \in W$ is the longest element and $T = \{ws w^{-1} \mid s \in S, w \in W\}$ is the set of all reflections in W (see [7, 5.3.1(a)]). Let $W' \subseteq W$ be a subgroup generated by reflections and let $T' = W' \cap T$. Let ε' be the sign character of W' . By a result due to Macdonald (see [7, 5.2.11]), there is a unique $\chi \in \text{Irr}(W)$ such that $\mathbf{b}_\chi = |T'|$ and

$$\text{Ind}_{W'}^W(\varepsilon') = \chi + \text{combination of various } \psi \in \text{Irr}(W) \text{ such that } \mathbf{b}_\psi > \mathbf{b}_\chi.$$

We shall denote this character by

$$\chi := \mathbf{j}_{W'}^W(\varepsilon').$$

This “j-induction” can be used to systematically construct all the irreducible characters of W of type A_{n-1} , B_n and D_n ; see [7, Chap. 5].

Example 3.2. Let $n \geq 1$ and $W = \mathfrak{S}_n$ be the symmetric group, where the generators are given by the basic transpositions $s_i = (i+1)$ for $1 \leq i \leq n-1$. (We also set $\mathfrak{S}_0 = \{1\}$.) It is well-known that the irreducible characters of \mathfrak{S}_n are parametrized by the partitions of n ; we write this as

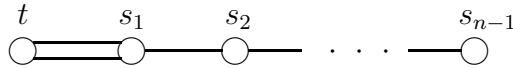
$$\text{Irr}(\mathfrak{S}_n) = \{\chi^\alpha \mid \alpha \vdash n\}.$$

This labelling is determined as follows; see, for example, [7, 5.4.7]. Given a partition $\alpha \vdash n$, we denote by α^* denote the transpose partition. Let $\mathfrak{S}_{\alpha^*} \subseteq \mathfrak{S}_n$ be the corresponding Young subgroup; we have $\mathfrak{S}_{\alpha^*} \cong \mathfrak{S}_{\alpha_1^*} \times \mathfrak{S}_{\alpha_2^*} \times \dots \times \mathfrak{S}_{\alpha_k^*}$, where $\alpha_1^*, \alpha_2^*, \dots, \alpha_k^*$ are the parts of α^* . Let ε_{α^*} be the sign character of \mathfrak{S}_{α^*} . Then

$$\chi^\alpha = \mathbf{j}_{\mathfrak{S}_{\alpha^*}}^{\mathfrak{S}_n}(\varepsilon_{\alpha^*}) \quad \text{and} \quad \mathbf{b}_{\chi^\alpha} = n(\alpha) := \sum_{1 \leq i \leq l} (i-1)\alpha_i$$

where $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_l \geq 0)$.

Example 3.3. Let $n \geq 1$ and \tilde{W}_n be a Coxeter group of type B_n , with generators $\{t, s_1, s_2, \dots, s_{n-1}\}$ and diagram given as follows.



(We also set $\tilde{W}_0 = \{1\}$.) The irreducible characters of \tilde{W}_n are parametrised by pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$. We write this as

$$\text{Irr}(\tilde{W}_n) = \{\tilde{\chi}^{(\alpha, \beta)} \mid (\alpha, \beta) \vdash n\}.$$

For $(\alpha, \beta) \vdash n$, there is a reflection subgroup $\tilde{W}_{\alpha, \beta} \subseteq \tilde{W}_n$ of type

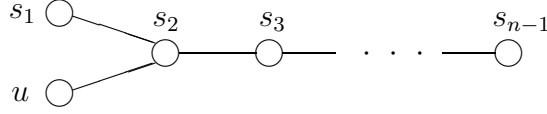
$$D_{\alpha_1} \times D_{\alpha_2} \times \dots \times D_l \times B_{\beta_1} \times B_{\beta_2} \times \dots \times B_{\beta_k}$$

where $\alpha_1, \alpha_2, \dots, \alpha_l$ are the parts of α and $\beta_1, \beta_2, \dots, \beta_k$ are the parts of β . Let $\tilde{\varepsilon}_{\alpha, \beta}$ be the sign character on $\tilde{W}_{\alpha, \beta}$. Then, by [7, 5.5.1, 5.5.3], we have

$$\tilde{\chi}^{(\alpha, \beta)} = \mathbf{j}_{\tilde{W}_{\alpha, \beta}}^{\tilde{W}_n}(\tilde{\varepsilon}_{\alpha, \beta}) \quad \text{and} \quad \mathbf{b}_{\tilde{\chi}^{(\alpha, \beta)}} = 2n(\alpha) + 2n(\beta) + |\beta|.$$

Note also that $\tilde{W}_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ and there is a corresponding description of $\text{Irr}(\tilde{W}_n)$ in terms of Clifford theory; see [7, 5.5.6].

Example 3.4. Let $n \geq 2$ and W_n be a Coxeter group of type D_n , with generators u, s_1, \dots, s_{n-1} and diagram given as follows.



Let $w \mapsto w^\diamond$ be defined by $u^\diamond = s_1$, $s_1^\diamond = u$ and $s_i^\diamond = s_i$ for $2 \leq i \leq n-1$. Then we can identify the semidirect product $W_n \rtimes \langle \diamond \rangle$ (see Remark 2.4) with a Coxeter group \tilde{W}_n of type B_n , with generators t, s_1, \dots, s_{n-1} and diagram as in Example 3.3. We have an embedding $W_n \hookrightarrow \tilde{W}_n$ given by the map

$$u \mapsto ts_1t, \quad s_1 \mapsto s_1, \quad s_2 \mapsto s_2, \quad \dots, \quad s_{n-1} \mapsto s_{n-1}.$$

Under this identification, we have $w^\diamond = twt$ for all $w \in W_n$. (Thus, the generator t is the “additional” element denoted by γ in Remark 2.4; by convention, we also set $W_0 = W_1 = \{1\}$, where $\tilde{W}_0 = \{1\}$ and $\tilde{W}_1 = \{1, t\}$.) This provides a convenient setting for classifying the irreducible characters of W_n . Given $(\alpha, \beta) \vdash n$, we denote by $\chi^{[\alpha, \beta]}$ the restriction of $\tilde{\chi}^{(\alpha, \beta)} \in \text{Irr}(\tilde{W}_n)$ to W_n . Then we have (see [7, 5.6.1, 5.6.2]):

- (a) If $\alpha \neq \beta$, then $\chi^{[\alpha, \beta]} = \chi^{[\beta, \alpha]} \in \text{Irr}(W_n)$. We have

$$\mathbf{b}_{\chi^{[\alpha, \beta]}} = 2n(\alpha) + 2n(\beta) + \min\{|\alpha|, |\beta|\}.$$

- (b) If $\alpha = \beta$, then $\chi^{[\alpha, \beta]} = \chi^{[\alpha, +]} + \chi^{[\alpha, -]}$ where $\chi^{[\alpha, +]}$, $\chi^{[\alpha, -]}$ are distinct irreducible characters of W_n . We have

$$\mathbf{b}_{\chi^{[\alpha, +]}} = \mathbf{b}_{\chi^{[\alpha, -]}} = 4n(\alpha) + n/2.$$

Furthermore, all irreducible characters of W_n arise in this way. Of course, the second case can only occur if n is even. In this case, the two characters $\chi^{[\alpha, \pm]}$ are explicitly given as follows; see [13, 4.6.2]. Let

$$H_n^+ = \langle s_1, s_2, \dots, s_{n-1} \rangle \quad \text{and} \quad H_n^- = \langle u, s_2, \dots, s_{n-1} \rangle.$$

Both H_n^+, H_n^- are isomorphic to \mathfrak{S}_n . Let $\alpha \vdash n/2$ and $\mathfrak{S}_{2\alpha^*}$ be the corresponding Young subgroup in \mathfrak{S}_n where $2\alpha^*$ denotes the partition of n obtained by multiplying all parts of α by 2. We have corresponding subgroups $H_{2\alpha^*}^+ \subseteq H_n^+$ and $H_{2\alpha^*}^- \subseteq H_n^-$. Then

$$\chi^{[\alpha, +]} = \mathbf{j}_{H_{2\alpha^*}^+}^{W_n}(\varepsilon_{2\alpha^*}^+) \quad \text{and} \quad \chi^{[\alpha, -]} = \mathbf{j}_{H_{2\alpha^*}^-}^{W_n}(\varepsilon_{2\alpha^*}^-)$$

where $\varepsilon_{2\alpha^*}^+$ denotes the sign character of $H_{2\alpha^*}^+$ and $\varepsilon_{2\alpha^*}^-$ denotes the sign character of $H_{2\alpha^*}^-$. (This is also discussed in [7, §5.6] but [7, 5.6.3] has to be reformulated as above.)

We take this occasion to correct an error in [7]. (This will actually be essential for the proof of the strengthened “branching rule” in Corollary 3.8.) Let ε be the sign character of W_n . In [7, Rem. 5.6.5], it is stated that $\chi^{[\alpha, +]} \otimes \varepsilon = \chi^{[\alpha^*, +]}$, where α^* denotes the conjugate partition. This can be easily seen to be wrong already in small examples. The correct statement is as follows.

Lemma 3.5. *Assume that $n \geq 2$ is even and let $\alpha \vdash n/2$.*

- (a) *If $n/2$ is even, then $\chi^{[\alpha, +]} \otimes \varepsilon = \chi^{[\alpha^*, +]}$ and $\chi^{[\alpha, -]} \otimes \varepsilon = \chi^{[\alpha^*, -]}$.*
(b) *If $n/2$ is odd, then*

$$\chi^{[\alpha, +]} \otimes \varepsilon = \chi^{[\alpha^*, -]} \quad \text{and} \quad \chi^{[\alpha, -]} \otimes \varepsilon = \chi^{[\alpha^*, +]}.$$

(c) Let $\sigma_{n/2} := s_1 s_3 \cdots s_{n-1} \in W_n$. (Note that $\sigma_{n/2}$ is a product of $n/2$ pairwise commuting generators s_i .) Then

$$\chi^{[\alpha,+]}(\sigma_{n/2}) - \chi^{[\alpha,-]}(\sigma_{n/2}) = (-1)^{n/2} 2^{n/2} \chi^\alpha(1)$$

where χ^α denotes the irreducible character of $\mathfrak{S}_{n/2}$ labelled by α .

Proof. In [7, 10.4.6] (see also [23, Thm. 5.1]), we find the definition of a collection of irreducible characters of W_n , which we denote here by $\{\psi^{[\alpha,\pm]} \mid \alpha \vdash n/2\}$, such that

$$\chi^{[\alpha,+]} + \chi^{[\alpha,-]} = \psi^{[\alpha,+]} + \psi^{[\alpha,-]} \quad \text{for all } \alpha \vdash n/2;$$

furthermore, it is shown there that

$$\psi^{[\alpha,+]}(\sigma_{n/2}) - \psi^{[\alpha,-]}(\sigma_{n/2}) = 2^{n/2} \chi^\alpha(1).$$

Note that this identity allows us to distinguish $\psi^{[\alpha,+]}$, $\psi^{[\alpha,-]}$ one from another. Tensoring with ε , we obtain

$$(\psi^{[\alpha,+]} \otimes \varepsilon)(\sigma_{n/2}) - (\psi^{[\alpha,-]} \otimes \varepsilon)(\sigma_{n/2}) = \varepsilon(\sigma_{n/2}) 2^{n/2} \chi^\alpha(1) = (-1)^{n/2} 2^{n/2} \chi^\alpha(1).$$

Now, by [7, 5.5.6], we have $\tilde{\chi}^{(\alpha,\alpha)} \otimes \tilde{\varepsilon} = \tilde{\chi}^{(\alpha^*,\alpha^*)}$ where $\tilde{\varepsilon}$ is the sign character of \tilde{W}_n (an extension of ε). This already implies that

$$\psi^{[\alpha,+]} \otimes \varepsilon = \psi^{[\alpha^*,\pm]} \quad \text{and} \quad \psi^{[\alpha,-]} \otimes \varepsilon = \psi^{[\alpha^*,\mp]}.$$

Comparing with the identity

$$\psi^{[\alpha^*,+]}(\sigma_{n/2}) - \psi^{[\alpha^*,-]}(\sigma_{n/2}) = 2^{n/2} \chi^{\alpha^*}(1) = 2^{n/2} \chi^{\alpha^*}(1),$$

we conclude that the desired description of the effect of tensoring with ε holds for the characters $\psi^{[\alpha,\pm]}$. Once this is shown, we can proceed as follows. By the computation in [7, 10.4.10], we have

$$\left\langle \text{Ind}_{H_{2\alpha^*}^+}^{W_n} (1_{2\alpha^*}), \psi^{[\alpha^*,+]} - \psi^{[\alpha^*,-]} \right\rangle_{W_n} = 1,$$

where $1_{2\alpha^*}$ stands for the trivial character of $H_{2\alpha^*}^+$. Consequently, we also have

$$\left\langle \text{Ind}_{H_{2\alpha^*}^+}^{W_n} (\varepsilon_{2\alpha^*}), \psi^{[\alpha^*,+]} \otimes \varepsilon - \psi^{[\alpha^*,-]} \otimes \varepsilon' \right\rangle_{W_n} = 1,$$

where $\varepsilon_{2\alpha^*}$ is the sign character of $H_{2\alpha^*}^+$. Comparing with the definition of $\chi^{[\alpha,\pm]}$ in Example 3.4, we conclude that we must have

$$\chi^{[\alpha,+]} = \psi^{[\alpha^*,+]} \otimes \varepsilon \quad \text{and} \quad \chi^{[\alpha,-]} = \psi^{[\alpha^*,-]} \otimes \varepsilon$$

for all $\alpha \vdash n/2$. This yields (a), (b), (c). \square

Remark 3.6. Assume that $n \geq 2$ is even and let $\sigma_{n/2} = s_1 s_3 \cdots s_{n-1} \in W_n$ as above. Let C_0 be the conjugacy class of $\sigma_{n/2}$ in W_n . By [7, Prop. 3.4.12], we have $tC_0 t \neq C_0$ and $\{C_0, tC_0 t\}$ is the only pair of conjugacy classes of involutions with this property. Furthermore, a direct computation shows (see also the formula in [23, 4.3]):

$$|C_{\mathfrak{S}_n}(\sigma_{n/2})| = 2^{n/2}(n/2)! \quad \text{and} \quad |C_{W_n}(\sigma_{n/2})| = 2^n(n/2)!.$$

We can now state the following strengthening of the induction formula in [7, 6.4.9].

Proposition 3.7. *Assume that $n \geq 2$ is even. Let $r \in \{2, 4, \dots, n\}$ and consider the parabolic subgroup $W' = W_{n-r} \times H_r$ where $W_{n-r} = \langle u, s_1, \dots, s_{n-r-1} \rangle$ (type D_{n-r}) and $H_r = \langle s_{n-r+1}, \dots, s_{n-1} \rangle \cong \mathfrak{S}_r$. Let $\alpha' \vdash (n-r)/2$ and denote by ε_r the sign character on the factor H_r . Then*

$$\text{Ind}_{W'}^{W_n}(\chi^{[\alpha', +]} \boxtimes \varepsilon_r) = \sum_{\alpha} \chi^{[\alpha, +]} + \text{“further terms”},$$

where the sum runs over all partitions $\alpha \vdash n/2$ whose Young diagram is obtained from that of α' by adding $r/2$ boxes, with no two boxes in the same row; the expression “further terms” stands for a sum of various $\chi \in \text{Irr}(W_n)$ which can be extended to \tilde{W}_n .

Proof. By [7, Prop. 6.4.9] (and its proof), we already know that

$$\text{Ind}_{W'}^{W_n}(\chi^{[\alpha', +]} \boxtimes \varepsilon_r) = \sum_{\alpha} \chi^{[\alpha, \mu_{\alpha}]} + \text{“further terms”},$$

$$\text{Ind}_{W'}^{W_n}(\chi^{[\alpha', -]} \boxtimes \varepsilon_r) = \sum_{\alpha} \chi^{[\alpha, -\mu_{\alpha}]} + \text{“further terms”},$$

where the sums run over all partitions $\alpha \vdash n/2$ as above and where $\mu_{\alpha} \in \{\pm 1\}$. So it remains to determine the signs μ_{α} . First note that the two “further terms” must be equal since the above induced characters are conjugate to each other under t . Hence, we have

$$\text{Ind}_{W'}^{W_n}(\chi^{[\alpha', +]} \otimes \varepsilon_r) - \text{Ind}_{W'}^{W_n}(\chi^{[\alpha', -]} \otimes \varepsilon_r) = \sum_{\alpha} (\chi^{[\alpha, \mu_{\alpha}]} - \chi^{[\alpha, -\mu_{\alpha}]})$$

where the sum runs over all partitions $\alpha \vdash n/2$ as above. To determine the signs, we evaluate both sides of this identity on the special element $\sigma_{n/2}$. Let C_0 denote the conjugacy class of $\sigma_{n/2}$. We have $\sigma_{n/2} \in W'$ and so $C_0 \cap W' \neq \emptyset$; furthermore, $C_0 \cap W'$ can only contain elements $w \in W'$ such that w, twt are not conjugate (see Remark 3.6). Consequently, $C_0 \cap W'$ is just the conjugacy class of W' containing $\sigma_{n/2}$. Now note that

$$\sigma_{n/2} = \sigma_{(n-r)/2} \times s_{n-r+1} s_{n-r+3} \cdots s_{n-1} \in W' = W_{n-2} \times H_r.$$

This yields

$$(\chi^{[\alpha', \pm]} \boxtimes \varepsilon_r)(\sigma_{n/2}) = \chi^{[\alpha', \pm]}(\sigma_{(n-r)/2}) \varepsilon_r(s_{n-r+1} s_{n-r+3} \cdots s_{n-1}) = (-1)^{r/2} \chi^{[\alpha', \pm]}(\sigma_{(n-r)/2})$$

and so

$$\text{Ind}_{W'}^{W_n}(\chi^{[\alpha', \pm]} \boxtimes \varepsilon_r)(\sigma_{n/2}) = (-1)^{r/2} \frac{|C_{W_n}(\sigma_{n/2})|}{|C_{W'}(\sigma_{n/2})|} \chi^{[\alpha', \pm]}(\sigma_{(n-r)/2}).$$

Furthermore, by the formulae in Remark 3.6, we have

$$\frac{|C_{W_n}(\sigma_{n/2})|}{|C_{W'}(\sigma_{(n-2)/2})|} = [\mathfrak{S}_{n/2} : \mathfrak{S}_{(n-r)/2} \times \mathfrak{S}_{r/2}] 2^{r/2}.$$

Thus, using also Lemma 3.5 (applied to W_{n-r}), we conclude that

$$\begin{aligned} \text{Ind}_{W'}^{W_n}(\chi^{[\alpha', +]} \otimes \varepsilon_r)(\sigma_{n/2}) - \text{Ind}_{W'}^{W_n}(\chi^{[\alpha', -]} \otimes \varepsilon_r)(\sigma_{n/2}) \\ = (-1)^{n/2} 2^{n/2} [\mathfrak{S}_{n/2} : \mathfrak{S}_{(n-r)/2} \times \mathfrak{S}_{r/2}] \chi^{\alpha'}(1). \end{aligned}$$

On the other hand, Lemma 3.5 applied to W_n yields that

$$\sum_{\alpha} (\chi^{[\alpha, \mu_{\alpha}]}(\sigma_{n/2}) - \chi^{[\alpha, -\mu_{\alpha}]}(\sigma_{n/2})) = (-1)^{n/2} 2^{n/2} \sum_{\alpha} \mu_{\alpha} \chi^{\alpha}(1),$$

where the sum runs over all partitions $\alpha \vdash n/2$ whose Young diagram is obtained from that of α' by adding $r/2$ boxes, with no two boxes in the same row. Now, by “Pieri’s Rule” for the characters of $\mathfrak{S}_{n/2}$ (see [7, 6.1.7]), we have

$$[\mathfrak{S}_{n/2} : \mathfrak{S}_{(n-r)/2} \times \mathfrak{S}_{r/2}] \chi^{\alpha'}(1) = \sum_{\alpha} \chi^{\alpha}(1)$$

where the sum runs over all α as above. Hence, we conclude that

$$\begin{aligned} (-1)^{n/2} \left(\text{Ind}_{W'}^{W_n} (\chi^{[\alpha', +]}) (\sigma_{n/2}) - \text{Ind}_{W'}^{W_n} (\chi^{[\alpha', -]}) (\sigma_{n/2}) \right) &= 2^{n/2} \left(\sum_{\alpha} \chi^{\alpha}(1) \right) \\ &\geq 2^{n/2} \left(\sum_{\alpha} \mu_{\alpha} \chi^{\alpha}(1) \right) = (-1)^{n/2} \left(\sum_{\alpha} (\chi^{\alpha, \mu_{\alpha}} (\sigma_{n/2}) - \chi^{[\alpha, -\mu_{\alpha}]} (\sigma_{n/2})) \right). \end{aligned}$$

Since the left hand side equals the right hand side, the inequality must be an equality which means that $\mu_{\alpha} = 1$ for all $\alpha \vdash n/2$, as desired. \square

Taking the special case $r = 2$, we obtain the following strengthened version of the “branching rule” for type D_n .

Corollary 3.8. *Assume that $n \geq 2$ is even. Consider the parabolic subgroup $W' = W_{n-2} \times H_2$ where $W_{n-2} = \langle u, s_1, \dots, s_{n-3} \rangle$ (type D_{n-2}) and $H_2 = \langle s_{n-1} \rangle$. Let $\alpha' \vdash (n-2)/2$ and denote by ε_1 the sign character on the factor H_2 . Then*

$$\text{Ind}_{W'}^{W_n} (\chi^{[\alpha', +]} \boxtimes \varepsilon_1) = \sum_{\alpha} \chi^{[\alpha, +]} + \text{“further terms”},$$

where the sum runs over all partitions $\alpha \vdash n/2$ such that α is obtained by increasing one part of α' by 1; the expression “further terms” stands for a sum of various $\chi \in \text{Irr}(W_n)$ which can be extended to W_n . In particular,

$$\left\langle \text{Ind}_{W'}^{W_n} (\chi^{[\alpha', +]} \boxtimes \varepsilon_1), \chi^{[\alpha, -]} \right\rangle_{W_n} = 0 \quad \text{for all } \alpha \vdash n/2.$$

Finally, we are able to describe the decomposition of Kottwitz’ characters Υ_w^1 and Υ_w° for W_n into irreducible characters.

3.9. Assume that $n \geq 2$ is even. Consider the element $\sigma_{n/2} \in W_n$ in Remark 3.6. Let $\Upsilon_{\sigma_{n/2}}^1$ be the character of the split version of Kottwitz’ involution module for W_n ; see Remark 2.3. By [11, §3.3], we have

$$\Upsilon_{\sigma_{n/2}}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha, \nu_{\alpha}]} \quad \text{and} \quad \Upsilon_{t\sigma_{n/2}t}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha, -\nu_{\alpha}]}$$

where $\nu_{\alpha} \in \{\pm 1\}$ for all $\alpha \vdash n/2$. But note that these signs have not been determined in [11]. Using an inductive argument based on Corollary 3.8, it is shown in [1, Prop. 7.4] that $\nu_{\alpha} = 1$ for all $\alpha \vdash n/2$. Thus, we have

$$\Upsilon_{\sigma_{n/2}}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha, +]} \quad \text{and} \quad \Upsilon_{t\sigma_{n/2}t}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha, -]}.$$

3.10. A complete set of representatives of the conjugacy classes of involutions in \tilde{W}_n is given as follows. Let l, j be non-negative integers such that $l + 2j \leq n$. Then set

$$\sigma_{l,j} := t_1 \cdots t_l s_{l+1} s_{l+3} \cdots s_{l+2j-1} \in \tilde{W}_n,$$

where $t_1 := t$ and $t_i := s_{i-1} t_{i-1} s_{i-1}$ for $2 \leq i \leq n$. Note that $\sigma_{l,j}$ is the longest element in a parabolic subgroup of \tilde{W}_n of type $B_l \times A_1 \times \cdots \times A_1$, where the A_1 factor is repeated j times. In particular, $\sigma_{l,j}$ is central in this parabolic subgroup and $\sigma_{l,j}$ has minimal length in its conjugacy class; see also [7, 3.2.10]. Every involution in \tilde{W}_n is conjugate to exactly one of the elements $\sigma_{l,j}$. Note that $\sigma_{l,j} \in W_n$ if and only if l is even. Furthermore, if n is odd, then every involution in W_n is conjugate (in W_n) to exactly one of the elements $\sigma_{l,j}$ where l is even. Assume now that n is even. Then

$$\sigma_{0,n/2} = s_1 s_3 \cdots s_{n-1} \in W_n$$

is the element already introduced in Remark 3.6. Let C_0 be the conjugacy class of $\sigma_{0,n/2}$ in W_n . Recall from Remark 3.6 that $C_0^\circ \neq C_0$ and that $\{C_0, C_0^\circ\}$ is the only pair of conjugacy classes of involutions in W_n with this property.

3.11. There is an alternative labelling of $\text{Irr}(\tilde{W}_n)$ in terms of Lusztig's "symbols". To describe this in more detail, let $(\alpha, \beta) \vdash n$ and consider the corresponding irreducible character $\tilde{\chi} = \tilde{\chi}^{(\alpha, \beta)} \in \text{Irr}(\tilde{W}_n)$; see Example 3.3. Choose $m \geq 1$ such that we can write

$$\alpha = (0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m) \quad \text{and} \quad \beta = (0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m).$$

As in [12, §1], [13, §4.18], we have a corresponding "symbol" with two rows of equal length

$$\Lambda_m(\tilde{\chi}) := \begin{pmatrix} \lambda_1, \lambda_2, \dots, \lambda_m \\ \mu_1, \mu_2, \dots, \mu_m \end{pmatrix}$$

where $\lambda_i := \alpha + i - 1$ and $\mu_i := \beta + i - 1$ for $1 \leq i \leq m$. We associate with $\tilde{\chi}$ the following invariants. First, we set

$$c(\tilde{\chi}) = c(\alpha, \beta) := \text{number of } i \in \{1, \dots, m\} \text{ such that } \mu_i \notin \{\lambda_1, \lambda_2, \dots, \lambda_m\},$$

$$d_0(\tilde{\chi}) = d_0(\alpha, \beta) := \beta_1 + \sum_{2 \leq i \leq m} \sup\{\alpha_{i-1}, \beta_i\}.$$

In particular, if $\alpha = \beta$, then $c(\alpha, \alpha) = 0$ and $d_0(\alpha, \alpha) = n/2$. Next, we set

$$\mathbf{a}_{\tilde{\chi}}^\diamond = \mathbf{a}_{(\alpha, \beta)}^\diamond := \sum_{1 \leq i \leq j \leq m} \min\{\lambda_i, \lambda_j\} + \sum_{1 \leq i \leq j \leq m} \min\{\mu_i, \mu_j\} \\ + \sum_{1 \leq i, j \leq m} \min\{\lambda_i, \mu_j\} - \frac{1}{6}m(m-1)(4m-5);$$

see [13, 4.6.3], [17, 22.14]. (Note that these definitions do not depend on the choice of m .)

(a) We say that $\tilde{\chi}$ is " \diamond -special" if $\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \cdots \leq \mu_m \leq \lambda_m$.

In particular, if $\tilde{\chi}$ is \diamond -special then either $\alpha = \beta$ or $|\beta| < |\alpha|$. Assume now that $\alpha \neq \beta$. Then $\chi = \chi^{[\alpha, \beta]} \in \text{Irr}(W_n)$ and the two extensions of χ to \tilde{W}_n are $\tilde{\chi}^{(\alpha, \beta)}$ and $\tilde{\chi}^{(\beta, \alpha)}$. Following Lusztig [15, 17.2], we say that $\chi^{(\alpha, \beta)}$ is the "*preferred extension*" of χ if the symbol $\Lambda_m(\tilde{\chi})$ has the following property: the smallest entry which appears in only one row appears in the lower row. Using also [13, 4.6.4], one easily verifies that:

(b) $\tilde{\chi}$ is \diamond -special if and only if $\mathbf{a}_{\tilde{\chi}}^{\diamond} = \mathbf{b}_{\chi}$ and $\tilde{\chi}$ is the preferred extension of χ .

This property played a role in the proof of the main result of [6] for type D_n , and it will also play a role in the proof of Proposition 4.9 below.

Proposition 3.12 (Kottwitz [11, §3.3 and §5.4]). *Let l, j be non-negative integers such that $l + 2j \leq n$; consider the corresponding involution $\sigma_{l,j} \in \tilde{W}_n$. Thus, if l is even, then $\sigma_{l,j} \in W_n$ in an “ordinary” involution; on the other hand, if l is odd, then $t\sigma_{l,j} \in W_n$ is a \diamond -twisted involution. Let*

$$\Upsilon'_{l,j} := \begin{cases} \Upsilon_{\sigma_{l,j}}^1 & \text{if } l \text{ is even,} \\ \Upsilon_{t\sigma_{l,j}}^{\diamond} & \text{if } l \text{ is odd,} \end{cases}$$

Then the following hold, where (α, β) is any pair of partitions such that $|\alpha| + |\beta| = n$ and, as before, $\chi^{[\alpha, \beta]}$ denotes the restriction of $\tilde{\chi}^{(\alpha, \beta)} \in \text{Irr}(\tilde{W}_n)$ to W_n .

(a) *We have $\langle \Upsilon'_{l,j}, \chi^{[\alpha, \beta]} \rangle_{W_n} = 0$ unless $\tilde{\chi}^{(\alpha, \beta)}$ is \diamond -special and $|\beta| = j$.*

(b) *If $\tilde{\chi}^{(\alpha, \beta)}$ is \diamond -special and $|\beta| = j$, then*

$$\langle \Upsilon'_{l,j}, \chi^{[\alpha, \beta]} \rangle_{W_n} = \binom{c(\alpha, \beta)}{j + l - d_0(\alpha, \beta)} \quad (\text{binomial coefficient});$$

in particular, the multiplicity is zero unless $d_0(\alpha, \beta) \leq j + l \leq d_0(\alpha, \beta) + c(\alpha, \beta)$.

(c) *If $j < n/2$, then $\langle \Upsilon'_{l,j}, \chi^{[\alpha, \beta]} \rangle_{W_n} = 0$ unless $\alpha \neq \beta$; if $2j = n$, then $\langle \Upsilon'_{l,j}, \chi^{[\alpha, \beta]} \rangle_{W_n} = 0$ unless $\alpha = \beta$.*

Proof. First assume that $j < n/2$; then $\sigma_{l,j}$ is not the special element $\sigma_{n/2}$ in Remark 3.6. The desired multiplicities in (a) and (b) are explicitly determined in [11, §3.3 and §5.4]. A similar argument applies to the case $j = n/2$ and $l = 0$. Using Example 3.4, we obtain

$$\langle \Upsilon'_{l,j}, \chi^{[\alpha, \alpha]} \rangle_{W_n} = \langle \Upsilon_{\sigma_{n/2}}^1, \chi^{[\alpha, +]} + \chi^{[\alpha, -]} \rangle_{W_n} = 1$$

in this case, as already mentioned in (3.9). \square

In order to prove Kottwitz' conjecture, our task now is to find similar formulae for the numbers of elements in the intersections of left cells with ordinary or \diamond -conjugacy classes of involutions in W_n .

Remark 3.13. Let \mathfrak{C} be a two-sided cell of W_n such that $\mathfrak{C}^{\diamond} \neq \mathfrak{C}$. By [13, 4.6.10, 5.25], this can only happen if n is even; in this case, we have

$$\text{Irr}(W_n \mid \mathfrak{C}) = \{\chi\} \quad \text{where} \quad \chi = \chi^{[\alpha, \pm]} \text{ for some } \alpha \vdash n/2.$$

Note that this implies that $[\Gamma]_1 = \chi$ for every left cell $\Gamma \subseteq \mathfrak{C}$. (More precisely, the above statement on $\text{Irr}(W_n \mid \mathfrak{C})$ implies that $[\Gamma]_1$ is a multiple of χ ; but then [13, 12.17] shows that $[\Gamma]_1$ is multiplicity-free.) Now let C be a \diamond -conjugacy class of \diamond -twisted involutions in W_n . In Remark 2.9, we have seen that $C \cap \Gamma = \emptyset$. We can now also show that

$$\langle \Upsilon_w^{\diamond}, [\Gamma]_1 \rangle_{W_n} = 0 \quad \text{for } w \in C.$$

Indeed, we are in the case where $l \neq 1$ is odd in Proposition 3.12. Then the formulae show that all irreducible constituents of Υ_w^{\diamond} are of the form $\chi^{[\alpha, \beta]}$ where $\alpha \neq \beta$. Hence, we must have $\langle \Upsilon_w^{\diamond}, [\Gamma]_1 \rangle_{W_n} = 0$ since $[\Gamma]_1 = \chi^{[\alpha, \pm]}$ for some $\alpha \vdash n/2$.

4. THE EXTENDED IWAHORI–HECKE ALGEBRA

The aim of this section is to establish certain positivity results for leading coefficients of character values of Iwahori–Hecke algebras in the quasi-split case. The analogous results in the split case were shown by Lusztig [13, 7.1], [16, 3.14]. The arguments are similar in the quasi-split case but some additional care is needed in choosing the correct extensions of the characters in $\text{Irr}^\diamond(W)$. Then Proposition 4.11 formulates the main applications to type D_n . We shall need a number of results from Lusztig’s book [13] and [16] so, as in Remark 2.15, we assume that W is a Weyl group and that \diamond is “ordinary” in the sense of [13, 3.1]. Let $\tilde{W} = \langle W, \gamma \rangle$ be the semidirect product as in Remark 2.4.

4.1. We begin by recalling some results concerning the representation theory of the (split semisimple) algebra $\mathbf{H}_K = K \otimes_A \mathbf{H}$, where $K = \mathbb{Q}(v)$ is the field of fractions of A . (See (2.7) for the definition of \mathbf{H} .) Via the specialisation $v \mapsto 1$, we obtain a canonical bijection

$$\text{Irr}(W) \leftrightarrow \text{Irr}(\mathbf{H}_K), \quad \chi \leftrightarrow \chi_v;$$

see [13, 3.3]. We have $\chi_v(T_w) \in A$ for all $w \in W$. For $\chi \in \text{Irr}(W)$, we define

$$\mathbf{a}_\chi := \min\{i \geq 0 \mid v^i \chi_v(T_w) \in \mathbb{Z}[v] \text{ for all } w \in W\}.$$

Then there are well-defined integers $c_{w,\chi} \in \mathbb{Z}$ such that

$$v^{\mathbf{a}_\chi} \chi_v(T_w) \equiv (-1)^{\ell(w)} c_{w,\chi} \pmod{v\mathbb{Z}[v]} \quad \text{for all } w \in W.$$

These are Lusztig’s “leading coefficients of character values”; see [13, Chap. 5], [16]. Note that the sum of all terms $c_{w,\chi}^2$ ($w \in W$) is a strictly positive number. Consequently, there is a well-defined positive rational number f_χ such that

$$\sum_{w \in W} c_{w,\chi}^2 = \chi(1) f_\chi.$$

In fact, it turns out that $f_\chi > 0$ is an integer; see [13, 4.1]. We have the following relation; see [13, Cor. 5.8]:

$$\sum_{w \in \Gamma} c_{w,\chi}^2 = f_\chi \langle [\Gamma]_1, \chi \rangle_W \quad \text{for any left cell } \Gamma \text{ of } W.$$

In particular, if \mathfrak{C} is a two-sided cell of W and $c_{w,\chi} \neq 0$ for some $w \in \mathfrak{C}$, then $\chi \in \text{Irr}(W \mid \mathfrak{C})$. We will now have to consider “quasi-split” versions of these constructions.

4.2. As in [13, 3.3], we can define an extended algebra $\tilde{\mathbf{H}}$ with an A -basis $\{T_\sigma \mid \sigma \in \tilde{W}\}$. For this purpose, we define a function $L: \tilde{W} \rightarrow \mathbb{Z}$ by $L(\gamma^i w) = \ell(w)$ for any $w \in W_n$ and $i = 0, 1$; in particular, L is an extension of the length function ℓ on W . Note that $L(\gamma) = 0$ and $L(\gamma w) = L(w^{-1}\gamma)$ for all $w \in W$. Then the multiplication in $\tilde{\mathbf{H}}$ is given as follows:

$$\begin{aligned} T_\sigma T_{\sigma'} &= T_{\sigma\sigma'} \quad \text{if } \sigma, \sigma' \in \tilde{W} \text{ are such that } L(\sigma\sigma') = L(\sigma) + L(\sigma'), \\ T_s^2 &= T_1 + (v - v^{-1})T_s \quad \text{if } s \in S. \end{aligned}$$

Thus, \mathbf{H} can be identified with the A -submodule of $\tilde{\mathbf{H}}$ spanned by all T_w ($w \in W$); note also that $\tilde{\mathbf{H}} = \mathbf{H}$ if $\diamond = 1$. Let $\{\mathbf{C}_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of \mathbf{H} as

defined in (2.7). Following [16, 3.1(a)], we extend this to a basis of $\tilde{\mathbf{H}}$ by setting

$$\mathbf{C}_{\gamma w} = T_{\gamma w} + \sum_{y \in W, y < w} (-1)^{\ell(w) - \ell(y)} v^{\ell(w) - \ell(y)} P_{y,w} T_{\gamma y} \quad (w \in W).$$

Correspondingly, we also have notions of left, right and two-sided cells in \tilde{W} ; in order to avoid any confusion with the analogous notions for W itself, we shall call them the left, right and two-sided L -cells in \tilde{W} . One easily sees that we have the following relations between the cells in W and the L -cells in \tilde{W} .

- (a) If Γ is a left cell in W , then $\Gamma^+ := \Gamma \cup \gamma \Gamma$ is a left L -cell of \tilde{W} . All left L -cells of \tilde{W} arise in this way.
- (b) If Γ, Γ^+ are as in (a), then the corresponding characters of W, \tilde{W} are related by

$$[\Gamma^+]_1 = \text{Ind}_{\tilde{W}}^{\tilde{W}}([\Gamma]_1).$$

- (c) If \mathfrak{C} is a two-sided cell of W , then $\mathfrak{C}^+ := \mathfrak{C} \cup \gamma \mathfrak{C} \cup \mathfrak{C} \gamma \cup \gamma \mathfrak{C} \gamma$ is a two-sided L -cell of \tilde{W} . All two-sided L -cells of \tilde{W} arise in this way.

(See [14, §16], [16, 3.1]; a related setting is considered in [5, 2.4.9].)

4.3. We extend the constructions in (4.1) to $\tilde{\mathbf{H}}$. Let $K = \mathbb{Q}(v)$ be the field of fractions of A . Then $\tilde{\mathbf{H}}_K := K \otimes_A \tilde{\mathbf{H}}$ is a split semisimple algebra. Via the specialisation $v \mapsto 1$, we obtain a canonical bijection

$$\text{Irr}(\tilde{W}) \leftrightarrow \text{Irr}(\tilde{\mathbf{H}}_K), \quad \tilde{\chi} \leftrightarrow \tilde{\chi}_v$$

which is compatible with the restriction of characters from \tilde{W} to W on the one side, and with the restriction of characters from $\tilde{\mathbf{H}}$ to \mathbf{H} on the other side; see [13, 3.3]. We have $\tilde{\chi}_v(T_\sigma) \in A$ for all $\sigma \in \tilde{W}$. For $\tilde{\chi} \in \text{Irr}(\tilde{W})$, we define

$$\mathbf{a}_{\tilde{\chi}} := \min\{i \geq 0 \mid v^i \chi_v(T_\sigma) \in \mathbb{Z}[v] \text{ for all } \sigma \in \tilde{W}\}.$$

Then there are well-defined integers $c_{\sigma, \tilde{\chi}} \in \mathbb{Z}$ such that

$$v^{\mathbf{a}_{\tilde{\chi}}} \tilde{\chi}_v(T_\sigma) \equiv (-1)^{L(\sigma)} c_{\sigma, \tilde{\chi}} \pmod{v\mathbb{Z}[v]} \quad \text{for all } \sigma \in \tilde{W}.$$

Again the sum of all terms $c_{\sigma, \tilde{\chi}}^2$ ($\sigma \in \tilde{W}$) is a strictly positive number. Consequently, there is a well-defined positive rational number $f_{\tilde{\chi}}$ such that

$$\sum_{\sigma \in \tilde{W}} c_{\sigma, \tilde{\chi}}^2 = \tilde{\chi}(1) f_{\tilde{\chi}}.$$

The relation between $\mathbf{a}_{\tilde{\chi}}, f_{\tilde{\chi}}$ and the analogous invariants for the irreducible characters of W are given as follows; see [5, Prop. 2.4.14]. Let $\tilde{\chi} \in \text{Irr}(\tilde{W})$ and $\chi \in \text{Irr}(W)$ be such that χ occurs in the restriction of $\tilde{\chi}$ to W . Then

$$(a) \quad \mathbf{a}_{\tilde{\chi}} = \mathbf{a}_{\chi} \quad \text{and} \quad \tilde{\chi}(1) f_{\tilde{\chi}} = 2 \chi(1) f_{\chi}.$$

Assume now that $\tilde{\chi}$ is an extension of some $\chi = \chi^\diamond \in \text{Irr}(W)$. Then we have the following relation with the left cells of W ; see [13, Cor. 5.8]:

$$(b) \quad \sum_{w \in \Gamma} c_{\gamma w, \tilde{\chi}}^2 = f_{\chi} \langle [\Gamma]_1, \chi \rangle_W \quad \text{for any left cell } \Gamma \text{ of } W.$$

In particular, if \mathfrak{C} is a two-sided cell of W and $c_{\gamma w, \tilde{\chi}} \neq 0$ for some $w \in W$, then $\chi \in \text{Irr}(W \mid \mathfrak{C})$.

Remark 4.4. Let $\chi \in \text{Irr}(W)$ and assume that $\chi^\diamond \neq \chi$. Then $\tilde{\chi} := \text{Ind}_{\tilde{W}}^{\tilde{W}}(\chi) \in \text{Irr}(\tilde{W})$. Correspondingly, if V is an \mathbf{H}_K -module affording χ_v , then

$$\tilde{V} = \tilde{\mathbf{H}}_K \otimes_{\mathbf{H}_K} V$$

is a $\tilde{\mathbf{H}}_K$ -module affording $\tilde{\chi}_v$; note that $\tilde{\mathbf{H}}_K$ is free as an \mathbf{H}_K -module, with basis $\{T_1, T_\gamma\}$. One easily sees that this implies that

$$\tilde{\chi}_v(T_w) = \chi_v(T_w) + \chi_v^\diamond(T_w) \quad \text{and} \quad \tilde{\chi}_v(T_{\gamma w}) = 0 \quad \text{for all } w \in W.$$

Hence, we obtain that

$$c_{w, \tilde{\chi}} = c_{w, \chi} + c_{w, \chi^\diamond} \quad \text{and} \quad c_{\gamma w, \tilde{\chi}} = 0 \quad \text{for all } w \in W.$$

4.5. Recall from [13, 4.1] that $\chi \in \text{Irr}(W)$ is called “*special*” if $\mathbf{a}_\chi = \mathbf{b}_\chi$. By [13, 4.14.2, 5.25], the sets $\text{Irr}(W \mid \mathfrak{C})$ (where $\mathfrak{C} \subseteq W$ is a two-sided cell) are explicitly known; in particular, it is known that each set $\text{Irr}(W \mid \mathfrak{C})$ contains a unique special character.

Now let $\tilde{\mathfrak{C}}$ be a two-sided L -cell of \tilde{W} and define

$$\text{Irr}(\tilde{W} \mid \tilde{\mathfrak{C}}) := \{ \tilde{\chi} \in \text{Irr}(\tilde{W}) \mid \langle \text{Ind}_{\tilde{W}}^{\tilde{W}}(\chi), \tilde{\chi} \rangle_{\tilde{W}} \neq 0 \text{ for some } \chi \in \text{Irr}(W \mid \mathfrak{C}) \}$$

where \mathfrak{C} is a two-sided cell of W such that $\tilde{\mathfrak{C}} = \mathfrak{C}^+$. (One easily sees that this does not depend on the choice of \mathfrak{C} ; note that there only is a choice if $\mathfrak{C} \neq \mathfrak{C}^\diamond$.) Using (4.2), we see that we obtain a partition

$$\text{Irr}(\tilde{W}) = \coprod_{\tilde{\mathfrak{C}}} \text{Irr}(\tilde{W} \mid \tilde{\mathfrak{C}})$$

where $\tilde{\mathfrak{C}}$ runs over all two-sided L -cells in \tilde{W} . We now define what it means for an irreducible character $\tilde{\chi} \in \text{Irr}(\tilde{W})$ to be “ \diamond -special”.

- (a) Assume that the restriction of $\tilde{\chi}$ to W is not irreducible. Then we say that $\tilde{\chi}$ is \diamond -special if $\tilde{\chi} = \text{Ind}_{\tilde{W}}^{\tilde{W}}(\chi)$ for some special $\chi \in \text{Irr}(W)$. (Note that χ is special if and only if χ^\diamond is special.)
- (b) Otherwise, $\tilde{\chi}$ is the extension of some $\chi \in \text{Irr}(W)$. In this case, we say that $\tilde{\chi}$ is \diamond -special if χ is special and $\tilde{\chi}$ is the “*preferred extension*” of χ in the sense of Lusztig [15, 17.2].

With the above definitions, each set $\text{Irr}(\tilde{W} \mid \tilde{\mathfrak{C}})$ will also contain a unique \diamond -special character of \tilde{W} . Note also that, if W of type D_n and \diamond is as in Example 3.4, then these definitions are consistent with those in (3.11).

Example 4.6. Let $W = W_n$ be of type D_n . Let \mathfrak{C} be a two-sided cell of W_n and $\chi_0 \in \text{Irr}(W_n \mid \mathfrak{C})$ be the unique special character. Let Γ be a left cell contained in \mathfrak{C} . Then the following hold:

- (a) $[\Gamma]_1$ is multiplicity-free with exactly f_{χ_0} irreducible constituents (one of which is χ_0); furthermore, Γ contains exactly f_{χ_0} involutions of W_n . (See [13, 12.17].)

Now let \diamond be the non-trivial graph automorphism as in Example 3.4. We identify \tilde{W} with a group \tilde{W}_n of type B_n . First we note:

- (b) $L: \tilde{W}_n \rightarrow \mathbb{Z}$ is a weight function in the sense of Lusztig [17]. Thus, $\tilde{\mathbf{H}}$ is the generic Iwahori–Hecke algebra associated with \tilde{W}_n, L as in [17]; furthermore, the notions of left, right and two-sided L -cells of \tilde{W}_n in (4.2) correspond exactly to the analogous notions in [17].
- (c) We have $\mathbf{a}_{\tilde{\chi}} = \mathbf{a}_{\tilde{\chi}}^{\diamond}$ for all $\tilde{\chi} \in \text{Irr}(\tilde{W}_n)$. (See [17, 22.14].)

Let $\tilde{\mathfrak{C}}$ be a two-sided L -cell of \tilde{W}_n and $\tilde{\chi}_0 \in \text{Irr}(\tilde{W}_n | \tilde{\mathfrak{C}})$ be the unique \diamond -special character. Let $\tilde{\Gamma}$ be a left L -cell contained in $\tilde{\mathfrak{C}}$. Then we have, where $c(\tilde{\chi}_0)$ is defined in (3.11):

- (d) $[\tilde{\Gamma}]_1$ is multiplicity-free with exactly $f_{\tilde{\chi}_0} = 2^{c(\tilde{\chi}_0)}$ irreducible constituents (one of which is $\tilde{\chi}_0$); furthermore, $\tilde{\Gamma}$ contains exactly $f_{\tilde{\chi}_0}$ involutions of \tilde{W}_n .

Indeed, there are two cases. Assume first that $\tilde{\chi}_0$ is an extension of a special character $\chi_0 \in \text{Irr}(W_n)$. Let \mathfrak{C} be the left cell of W_n such that $\chi_0 \in \text{Irr}(W_n | \mathfrak{C})$. By (2.15)(c), we have $\text{Irr}(W_n | \mathfrak{C}) \subseteq \text{Irr}^{\diamond}(W_n)$. So (a) and (4.2)(b) imply that $[\tilde{\Gamma}]_1$ is multiplicity-free with exactly $2f_{\chi_0}$ irreducible constituents (one of which is $\tilde{\chi}_0$). By (4.3)(a), we have $f_{\tilde{\chi}_0} = 2f_{\chi_0}$. Now assume that $\tilde{\chi}_0$ is obtained by inducing a special $\chi_0 \in \text{Irr}(W_n)$ to \tilde{W}_n . Then n is even and $\chi_0 = \chi^{[\alpha, \pm]}$ for some $\alpha \vdash n/2$. Again, let \mathfrak{C} be the left cell of W_n such that $\chi_0 \in \text{Irr}(W_n | \mathfrak{C})$. Then $[\Gamma]_1 = \chi_0$ for any left cell $\Gamma \subseteq \mathfrak{C}$; see Remark 3.13. So (4.2)(b) implies that $[\tilde{\Gamma}]_1 = \tilde{\chi}_0 = \tilde{\chi}^{(\alpha, \alpha)}$ is irreducible. By (4.3)(b), we have $f_{\tilde{\chi}_0} = f_{\chi_0}$. In both cases, the equality $f_{\tilde{\chi}_0} = 2^{c(\tilde{\chi}_0)}$ follows from the explicit formula in [17, 22.14]; note that $c(\tilde{\chi}_0) = 0$ in the second case. This completes the proof of the statement concerning the decomposition of $[\tilde{\Gamma}]_1$. To prove the statement concerning the involutions in $\tilde{\Gamma}$, we argue as follows. By (b), we can apply [3, Theorem 1.1] to $\tilde{\mathbf{H}}$ which shows that the number of (ordinary) involutions in $\tilde{\Gamma}$ equals the number of irreducible constituents of $[\tilde{\Gamma}]_1$ (counting multiplicities). Since $[\tilde{\Gamma}]_1$ is multiplicity-free, this yields the desired statement.

4.7. Assume that (W, S, \diamond) arises from a connected reductive algebraic group \mathbf{G} and a Frobenius map F , corresponding to some \mathbb{F}_q -rational structure on \mathbf{G} . Thus, W is the Weyl group of \mathbf{G} with respect to an F -stable maximal torus which is contained in an F -stable Borel subgroup of \mathbf{G} ; furthermore, $w \mapsto w^{\diamond}$ is the map induced by F on W . Let $G = \mathbf{G}^F$. For each $\chi \in \text{Irr}^{\diamond}(W)$, let $R_{\tilde{\chi}}$ be the corresponding “almost character” of G (see [13, 3.7]), where $\tilde{\chi} \in \text{Irr}(\tilde{W})$ is an extension of χ . (If we choose another extension $\tilde{\chi}'$ of χ , then $R_{\tilde{\chi}'} = \pm R_{\tilde{\chi}}$.) Let

$$\text{Uch}(G) = \{\rho \in \text{Irr}(G) \mid \langle R_{\tilde{\chi}}, \rho \rangle_G \neq 0 \text{ for some } \chi \in \text{Irr}^{\diamond}(W)\}$$

be the set of *unipotent characters* of G . For any two-sided cell \mathfrak{C} of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$, we denote by $\text{Uch}(G | \mathfrak{C})$ the set of all $\rho \in \text{Uch}(G)$ such that $\langle R_{\tilde{\chi}}, \rho \rangle_G \neq 0$ for some $\chi \in \text{Irr}(W | \mathfrak{C})$. By the “Disjointness Theorem” [13, 6.16], we obtain a partition

$$\text{Uch}(G) = \coprod_{\mathfrak{C}} \text{Uch}(G | \mathfrak{C})$$

where \mathfrak{C} runs over all two-sided cells of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$. All the multiplicities $\langle R_{\tilde{\chi}}, \rho \rangle_G$ are explicitly described by [13, Main Theorem 4.23]; this involves a certain Fourier matrix and a function $\Delta: \text{Uch}(G) \rightarrow \{\pm 1\}$.

To state the following result, we introduce the following notation. For $\chi \in \text{Irr}^\diamond(W)$, we set

$$c_{\gamma w, \tilde{\chi}} := (-1)^{\mathbf{a}_\chi + l(w)} c_{\gamma w, \tilde{\chi}} \quad \text{for all } w \in W,$$

where $\tilde{\chi} \in \text{Irr}(\tilde{W})$ is an extension of χ to \tilde{W} .

Proposition 4.8 (Lusztig [13, 7.1], [16]). *In the above setting, let \mathfrak{C} be a two-sided cell of W such that $\mathfrak{C}^\diamond = \mathfrak{C}$. Assume there exists some $\chi_0 \in \text{Irr}^\diamond(W)$ and an extension $\tilde{\chi}_0 \in \text{Irr}(\tilde{W})$ of χ_0 such that*

$$(*) \quad \Delta(\rho) \langle R_{\tilde{\chi}_0}, \rho \rangle_G > 0 \quad \text{for all } \rho \in \text{Uch}(G^F \mid \mathfrak{C}).$$

Then $c_{\gamma w, \tilde{\chi}_0}^ \geq 0$ for all $w \in \mathfrak{C}$; furthermore, $c_{\gamma w, \tilde{\chi}_0}^* > 0$ for all $w \in \mathfrak{C}$ such that w^\diamond, w^{-1} belong to the same left cell of W .*

Proof. First consider the inequality $c_{\gamma w, \tilde{\chi}_0}^* \geq 0$ for all $w \in \mathfrak{C}$. In [13, 7.1], this is proved assuming that F acts trivially on W and $\Delta(\rho) = 1$ for all $\rho \in \text{Uch}(G \mid \mathfrak{C})$. But the same proof gives the more general statement above. Let us briefly sketch the main ingredients. For each $\chi \in \text{Irr}^\diamond(W)$ (other than χ_0), let us fix some extension $\tilde{\chi} \in \text{Irr}(\tilde{W})$. Let $w \in \mathfrak{C}$ and consider the class function

$$R_{\gamma w} := \sum_{\chi \in \text{Irr}^\diamond(W)} c_{\gamma w, \tilde{\chi}} R_{\tilde{\chi}}$$

on G . (Note that this is independent of any choices.) Since $w \in \mathfrak{C}$, we have that $R_{\gamma w}$ is a linear combination of the unipotent characters in $\text{Uch}(G \mid \mathfrak{C})$; see [13, 5.2]. Since the functions $\{R_{\tilde{\chi}} \mid \chi \in \text{Irr}^\diamond(W)\}$ form an orthonormal system (see [13, 3.9]), we obtain

$$c_{\gamma w, \tilde{\chi}_0} = \sum_{\rho \in \text{Uch}(G \mid \mathfrak{C})} \langle R_{\tilde{\chi}_0}, \rho \rangle_G \langle R_{\gamma w}, \rho \rangle_G.$$

Now let $\rho \in \text{Uch}(G \mid \mathfrak{C})$ be such that the corresponding terms in the above sum are non-zero. Then [13, 6.19] shows that $(-1)^{\mathbf{a}_{\chi_0} + l(w)} = \Delta(\rho)$. Hence, we obtain

$$c_{\gamma w, \tilde{\chi}_0}^* = (-1)^{\mathbf{a}_{\chi_0} + l(w)} c_{\gamma w, \tilde{\chi}_0} = \sum_{\rho \in \text{Uch}(G \mid \mathfrak{C})} \Delta(\rho) \langle R_{\tilde{\chi}_0}, \rho \rangle_G \langle R_{\gamma w}, \rho \rangle_G.$$

Now, by the “Disjointness Theorem” [13, 6.17], $R_{\gamma w}$ is an actual character of G and so $\langle R_{\gamma w}, \rho \rangle_G \geq 0$ for all $\rho \in \text{Uch}(G \mid \mathfrak{C})$. Since $(*)$ is assumed to hold, we conclude that $c_{\gamma w, \tilde{\chi}_0} \geq 0$ for all $w \in \mathfrak{C}$.

Finally, assume that $w \in \mathfrak{C}$ is such that w^\diamond, w^{-1} belong to the same left cell of W . Then we must prove that $c_{\gamma w, \tilde{\chi}_0}^* \neq 0$. Now, the above expression for $c_{\gamma w, \tilde{\chi}_0}^*$ (together with $(*)$) shows that it will be sufficient to prove that there exists some $\rho \in \text{Uch}(G \mid \mathfrak{C})$ such that $\langle R_{\gamma w}, \rho \rangle_G \neq 0$. For this purpose, it is enough to show that $R_{\gamma w} \neq 0$. Furthermore, since the class functions $\{R_{\tilde{\chi}}\}$ are linearly independent, it will be sufficient to show that $c_{\gamma w, \tilde{\chi}} \neq 0$ for some $\chi \in \text{Irr}^\diamond(W)$. But this follows by an argument involving Lusztig’s asymptotic algebra $\tilde{\mathbf{J}}$; see [16, 3.1]. Indeed, this algebra has a basis $\{t_\sigma \mid \sigma \in \tilde{W}\}$ where the structure constants are integers. It is known that $\tilde{\mathbf{J}}$ is a “based ring” in the sense of [16, §1]; see [16, 3.1(j)]. This has several consequences. First of all, by [16, 3.1(k)], the elements σ, σ^{-1} belong to the same left L -cell in \tilde{W} if and only if $t_\sigma^2 \neq 0$. Furthermore, by [16, 1.2(b)], we have $t_\sigma^2 \neq 0$ if and only if some irreducible character of $\tilde{\mathbf{J}}$ has a non-zero

value on t_σ . Finally, by [16, 3.4(a), (e)], the leading coefficients $c_{\sigma, \tilde{\psi}}$ can be interpreted (up to signs) as the values of the irreducible characters of \tilde{J} on t_σ . Thus, we have:

$$\sigma, \sigma^{-1} \text{ belong to the same left } L\text{-cell} \quad \Leftrightarrow \quad c_{\sigma, \tilde{\psi}} \neq 0 \text{ for some } \tilde{\psi} \in \text{Irr}(\tilde{W}).$$

Now return to our element $w \in \mathfrak{C}$ such that w^\diamond, w^{-1} belong to the same left cell of W . Since \diamond permutes the left cells of W , we also have that $w, (w^\diamond)^{-1}$ belong to the same left cell of W . Consequently, since $(\gamma w)^{-1} = w^{-1}\gamma = \gamma(w^\diamond)^{-1}$, the elements $\gamma w, (\gamma w)^{-1}$ belong to the same left L -cell of \tilde{W} . So the above equivalence shows that there exists some $\tilde{\psi} \in \text{Irr}(\tilde{W})$ such that $c_{\gamma w, \tilde{\psi}} \neq 0$. But then Remark 4.4 implies that $\tilde{\psi}$ must be an extension of some $\psi \in \text{Irr}^\diamond(W)$, as required. \square

Proposition 4.9. *Assume that we are in the setting of (4.7). Let \mathfrak{C} be a two-sided cell of W such that $\mathfrak{C}^\diamond = \mathfrak{C}$. Let $\chi_0 \in \text{Irr}(W)$ be the unique special character in $\text{Irr}(W \mid \mathfrak{C})$; we have $\chi_0^\diamond = \chi_0$. Then condition $(*)$ in Proposition 4.8 holds if $\tilde{\chi}_0$ is the \diamond -special extension of χ_0 in the sense of (4.5).*

Proof. By standard reduction arguments, it is enough to prove this in the case where W is irreducible. If F acts trivially on W , then the multiplicity formula in [13, Main Theorem 4.23] shows that $\langle R_{\tilde{\chi}_0}, \rho \rangle_G = \Delta(\rho)$ for all $\rho \in \text{Uch}(G \mid \mathfrak{C})$. (The special character χ_0 corresponds to the pair $(1, 1)$ in the set $\mathcal{M}(\mathcal{G}_{\mathfrak{C}})$ where $\mathcal{G}_{\mathfrak{C}}$ is the finite group associated with \mathfrak{C} .) Hence, the assertion is clear in this case. So let us now assume that F does not act trivially on W . Then we only have 2 cases to consider:

- (a) W is of type E_6 or A_n and \diamond is given by conjugation with the longest element.
- (b) $W = W_n$ is of type D_n and \diamond is given as in Example 3.4.

Assume that we are in case (a). Then $\chi^\diamond = \chi$ for all $\chi \in \text{Irr}(W)$. Given χ , the “preferred extension” $\tilde{\chi}$ is determined by the condition that γw_0 acts as $(-1)^{\mathbf{a}_\chi}$ in a representation affording $\tilde{\chi}$. The assertion then follows from the description of the Fourier matrix in [13, 4.19] and the Δ -function in [13, p. 124]. Now assume that we are in case (b). The “preferred extensions” are described in (3.11). We have $\Delta(\rho) = 1$ for all $\rho \in \text{Uch}(G)$; see [13, 6.18.5, 6.19]. The assertion now follows from the description of the Fourier matrix in [13, 4.18]; see also [12, Theorem 3.15]. (This has also been discussed in some detail in the proof of [6, Theorem 5.1].) \square

Lemma 4.10 (“The basic identity”; cf. [1, §3]). *Let \mathfrak{C} be a two-sided cell of W such that $\mathfrak{C}^\diamond = \mathfrak{C}$. Let Γ be a left cell of W such that $\Gamma \subseteq \mathfrak{C}$ and C be a \diamond -conjugacy class of \diamond -twisted involutions of W . Then*

$$\langle [\Gamma]_1, \chi \rangle_W \sum_{w \in C \cap \mathfrak{C}} c_{\gamma w, \tilde{\chi}} = \chi(1) \sum_{w \in C \cap \Gamma} c_{\gamma w, \tilde{\chi}} \quad \text{for all } \chi \in \text{Irr}^\diamond(W),$$

where $\tilde{\chi} \in \text{Irr}(\tilde{W})$ denotes a fixed extension of $\chi \in \text{Irr}^\diamond(W)$ to \tilde{W} .

Proof. If $\diamond = 1$, then this is proved in [1, Lemma 3.1]. The general case is completely analogous. First, as in the proof of [1, Lemma 1.2], one verifies that $T_{s^\diamond} Z = Z T_s$ for all $s \in S$, where

$$Z := \sum_{w \in C} (-1)^{\ell(w)} T_w \in \mathbf{H}.$$

Consequently, the element $\tilde{Z} := \sum_{w \in C} (-1)^{\ell(w)} T_{\gamma w} \in \tilde{\mathbf{H}}$ lies in the centre of $\tilde{\mathbf{H}}$. Once this is established, the proof proceeds exactly as in [1, Lemma 3.1]. All the required properties of the leading coefficients $c_{\gamma w, \tilde{\chi}}$ and the structure constants of the Kazhdan–Lusztig basis of $\tilde{\mathbf{H}}$ hold by [16, 3.1–3.4]. \square

We now apply the above results to type D_n .

Proposition 4.11. *Let $W = W_n$ be of type D_n and \diamond be as in Example 3.4. We identify \tilde{W} with a group \tilde{W}_n of type B_n . Let \mathfrak{C} be a two-sided cell of W_n and $\chi_0 \in \text{Irr}(W_n)$ be the unique special character in $\text{Irr}(W_n \mid \mathfrak{C})$. Then the following hold.*

- (a) *Assume that $\mathfrak{C}^\diamond = \mathfrak{C}$ and let $\tilde{\chi}_0 \in \text{Irr}(\tilde{W}_n)$ be the preferred extension of $\chi_0 = \chi_0^\diamond$. Then*

$$c_{tw, \tilde{\chi}_0}^* = 1 \quad \text{for all } \diamond\text{-twisted involutions } w \in \mathfrak{C}.$$

furthermore, each left cell $\Gamma \subseteq \mathfrak{C}$ contains exactly f_{χ_0} \diamond -twisted involutions.

- (b) *Let C be a \diamond -conjugacy class of \diamond -twisted involutions in W_n . Then*

$$|C \cap \mathfrak{C}| = \chi_0(1) |C \cap \Gamma| \quad \text{for any left cell } \Gamma \subseteq \mathfrak{C}.$$

Proof. (a) Let $w \in \mathfrak{C}$ be a \diamond -twisted involution. Then $w^\diamond = w^{-1}$ belong to the same left cell. By Propositions 4.8 and 4.9, we conclude that $c_{tw, \tilde{\chi}_0}^* > 0$. In order to show that $c_{tw, \tilde{\chi}_0}^* = 1$, we use a counting argument. Let $\Gamma \subseteq \mathfrak{C}$ be any left cell. Since $c_{tw, \tilde{\chi}_0}^* > 0$ for all \diamond -twisted involutions $w \in \Gamma$, we have

$$(\text{number of } \diamond\text{-twisted involutions in } \mathfrak{C}) \leq \sum_{w \in \mathfrak{C}} c_{tw, \tilde{\chi}_0}^2,$$

with equality only if $c_{tw, \tilde{\chi}_0}^* = 1$ for all \diamond -twisted involutions $w \in \Gamma$. By (4.3)(b), the right hand side equals $f_{\chi_0} \langle [\Gamma]_1, \chi_0 \rangle_{W_n}$. By Example 4.6(a), we have $\langle [\Gamma]_1, \chi_0 \rangle_{W_n} = 1$ and so

$$(\text{number of } \diamond\text{-twisted involutions in } \Gamma) \leq f_{\chi_0}.$$

Now consider the left L -cell $\Gamma^+ = \Gamma \cup t\Gamma$ of \tilde{W}_n . Using Example 4.6(a) and Remark 2.4, the above inequality can be rephrased as:

$$(\text{number of ordinary involutions in } \Gamma^+) \leq 2f_{\chi_0},$$

where equality holds if and only if equality holds in all the previous inequalities. But then Example 4.6(d) shows that all the previous inequalities must be equalities; note that $f_{\tilde{\chi}_0} = 2f_{\chi_0}$ in this case. In particular, $c_{tw, \tilde{\chi}_0}^* = 1$ for all \diamond -twisted involutions $w \in \Gamma$. It also follows that the number of these \diamond -twisted involutions equals f_{χ_0} . Thus, (a) is proved.

(b) If $\mathfrak{C}^\diamond \neq \mathfrak{C}$, then both sides of the equality are zero by Remark 2.9. So let us now assume that $\mathfrak{C}^\diamond = \mathfrak{C}$ and let $\tilde{\chi}_0$ be as in (a). By Example 4.6(a), we have $\langle [\Gamma]_1, \chi_0 \rangle_{W_n} = 1$. It remains to use the identity in Lemma 4.10. \square

5. TWISTED INVOLUTIONS IN TYPE D_n

Throughout this section we place ourselves in the setting of Example 3.4. Thus, $n \geq 2$ and $W = W_n$ is a Coxeter group of type D_n , with generators u, s_1, \dots, s_{n-1} . Let $w \mapsto w^\diamond$ be defined by $u^\diamond = s_1$, $s_1^\diamond = u$ and $s_i^\diamond = s_i$ for $2 \leq i \leq n-1$. We identify the semidirect

product $\tilde{W} = W \rtimes \langle \diamond \rangle$ with the Coxeter group \tilde{W}_n of type B_n with generators t, s_1, \dots, s_{n-1} as in Example 3.3. Recall that, under this identification, we have

$$w^\diamond = twt \quad \text{for all } w \in W_n.$$

Let $L: \tilde{W}_n \rightarrow \mathbb{Z}$ be defined as in (4.2). We have already noted in Example 4.6(c) that L is a weight function in the sense of Lusztig [17]; explicitly, we have

$$L(t) = 0 \quad \text{and} \quad L(s_1) = L(s_2) = \dots = L(s_{n-1}) = 1.$$

We shall consider the left, right and two-sided L -cells of \tilde{W}_n . In Theorem 5.3, we state a modified version of Kottwitz' conjecture for all (ordinary) conjugacy classes of involutions in \tilde{W}_n . This crucially relies on the construction of the modified involution module in Lemma 5.2. Then note that any involution in \tilde{W}_n is either an ordinary involution in W_n or corresponds to a \diamond -twisted involution in W_n . In Corollary 5.4, we will see that the modified version of Kottwitz' conjecture for \tilde{W}_n encapsulates both the split and the quasi-split version of Kottwitz' conjecture for W_n .

Throughout, it will be convenient to allow also the possibility that $n = 0, 1$, where $\tilde{W}_0 = \{1\}$, $\tilde{W}_1 = \{1, t\}$ and $W_0 = W_1 = \{1\}$.

Remark 5.1. We have the following relation between the length function ℓ on W_n and the length function $\tilde{\ell}$ on \tilde{W}_n . Let $w \in W_n$ and $1 \leq i \leq n-1$. Then we have

$$\ell(ws_i) < \ell(w) \iff \tilde{\ell}(ws_i) < \tilde{\ell}(w) \iff \tilde{\ell}(tws_i) < \tilde{\ell}(tw).$$

Indeed, first note that, by [7, 1.4.12], we have $\tilde{\ell}(w) = \ell(w) + \ell_t(w)$, where $\ell_t(w)$ denotes the number of occurrences of t in a reduced expression of w in terms of the generators of \tilde{W}_n . This implies the first equivalence. To prove the second equivalence, we distinguish two cases. Suppose first that $\tilde{\ell}(tw) > \tilde{\ell}(w)$. Now, if $\tilde{\ell}(ws_i) < \tilde{\ell}(w)$, then $\tilde{\ell}(tws_i) \leq \tilde{\ell}(w) < \tilde{\ell}(tw)$, as required. Conversely, assume that $\tilde{\ell}(tws_i) < \tilde{\ell}(tw)$. Since $\tilde{\ell}(tw) > \tilde{\ell}(w)$, this implies $\tilde{\ell}(tws_i) = \tilde{\ell}(w)$. So we must have $\tilde{\ell}(ws_i) < \tilde{\ell}(w)$ by [7, Lemma 1.2.6]. (Otherwise, we would have $tws_i = w$ and so t, s_i would be conjugate, a contradiction.) The argument for the case $\tilde{\ell}(tw) < \tilde{\ell}(w)$ is similar.

Lemma 5.2. *Let \mathcal{C} be an (ordinary) conjugacy class of involutions of \tilde{W}_n . Let \tilde{M} be a \mathbb{Q} -vector space with a basis $\{\tilde{a}_\sigma \mid \sigma \in \mathcal{C}\}$. Then \tilde{M} is a $\mathbb{Q}[\tilde{W}_n]$ -module, where the action is given by:*

$$\begin{aligned} t.\tilde{a}_\sigma &= \tilde{a}_{t\sigma t}, \\ s_i.\tilde{a}_\sigma &= \begin{cases} -\tilde{a}_\sigma & \text{if } s_i\sigma = \sigma s_i \text{ and } \tilde{\ell}(\sigma s_i) < \tilde{\ell}(\sigma), \\ \tilde{a}_{s_i\sigma s_i} & \text{otherwise,} \end{cases} \end{aligned}$$

for $1 \leq i \leq n-1$. Furthermore, let $\tilde{\Upsilon}_{\mathcal{C}}$ denote the character of \tilde{W}_n afforded by \tilde{M} .

- (a) If $\mathcal{C} \subseteq W_n$ and \mathcal{C} is a single conjugacy class in W_n , then the restriction of $\tilde{\Upsilon}_{\mathcal{C}}$ to W_n is Kottwitz' character $\Upsilon_{\mathcal{C}}^1$ for W_n (split case, see Remark 2.3).
- (b) If $\mathcal{C} \subseteq W_n$ and \mathcal{C} consists of two conjugacy classes in W_n , then n is even and the restriction of $\tilde{\Upsilon}_{\mathcal{C}}$ to W_n equals the sum of the two characters $\Upsilon_{\sigma_{n/2}}^1$ and $\Upsilon_{t\sigma_{n/2}t}^1$ in (3.9).

- (c) If $\mathcal{C} \subseteq W_n t$, then $C = \{t\sigma \mid \sigma \in \mathcal{C}\}$ is a \diamond -conjugacy class of \diamond -twisted involutions in W_n and the restriction of $\tilde{\Upsilon}_{\mathcal{C}}$ to W_n is Kottwitz' character $\Upsilon_{\mathcal{C}}^{\diamond}$ for W_n (quasi-split case).

Proof. If $\mathcal{C} \subseteq W_n$, then this result is contained in [6, Prop. 2.4], with the only difference that the length condition is expressed in terms of the length ℓ on W_n . But then the first equivalence in Remark 5.1 allows us to rewrite this condition as above. By [6, Rem. 2.2], the character of the restriction of \tilde{M} to W_n is $\Upsilon_{\mathcal{C}}^1$. This yields (a) if \mathcal{C} is a single conjugacy class in W_n ; otherwise, n must be even and $\mathcal{C} = C_0 \cup tC_0 t$ where C_0 is the conjugacy class of the element $\sigma_{n/2}$ in Remark 3.6; this yields (b).

Now assume that $\mathcal{C} \subseteq tW_n$. Then recall from Remark 2.4 that $C := \{t\sigma \mid \sigma \in \mathcal{C}\}$ is a \diamond -conjugacy class of \diamond -twisted involutions in W_n . Let M be a \mathbb{Q} -vector space with a basis $\{a_w \mid w \in C\}$. By [21, 7.1] (see also [19]), we already know that M is a $\mathbb{Q}[W_n]$ -module, where the action is given by the following formulae for any $s \in \{u, s_1, \dots, s_{n-1}\}$:

$$s.a_w = \begin{cases} -a_w & \text{if } s^\diamond w = ws \text{ and } \ell(ws) < \ell(w), \\ a_{s^\diamond w} & \text{otherwise.} \end{cases}$$

By (2.6), the character of W_n afforded by M is $\Upsilon_{\mathcal{C}}^{\diamond}$. Via the linear map $M \rightarrow \tilde{M}$, $a_w \mapsto \tilde{a}_{tw}$, we can transport this action to \tilde{M} . The action of W_n on \tilde{M} is given by the following formulae for any $s \in \{u, s_1, \dots, s_{n-1}\}$:

$$s.\tilde{a}_\sigma = \begin{cases} -\tilde{a}_\sigma & \text{if } s\sigma = \sigma s \text{ and } \ell(t\sigma s) < \ell(t\sigma), \\ \tilde{a}_{s\sigma s} & \text{otherwise.} \end{cases}$$

Now, by [20, 0.4], this action can be extended to \tilde{W}_n via the formulae:

$$\begin{aligned} t.\tilde{a}_\sigma &= \tilde{a}_{t\sigma t}, \\ s_i.\tilde{a}_\sigma &= \begin{cases} -\tilde{a}_\sigma & \text{if } s_i\sigma = \sigma s_i \text{ and } \ell(t\sigma s_i) < \ell(t\sigma), \\ \tilde{a}_{s_i\sigma s_i} & \text{otherwise;} \end{cases} \end{aligned}$$

Then Remark 5.1 shows that the conditions involving the length function ℓ on W_n can be rewritten in terms of the length function $\tilde{\ell}$ on W_n . \square

We can now state the main result of this section.

Theorem 5.3. *Let \mathcal{C} be any (ordinary) conjugacy class of involutions in \tilde{W}_n . Then*

$$\boxed{\langle \tilde{\Upsilon}_{\mathcal{C}}, [\tilde{\Gamma}]_1 \rangle_{\tilde{W}_n} = |\mathcal{C} \cap \tilde{\Gamma}| \quad \text{for any left } L\text{-cell } \tilde{\Gamma} \subseteq \tilde{W}_n;}$$

here, $\tilde{\Upsilon}_{\mathcal{C}}$ is the character of the $\mathbb{Q}[\tilde{W}_n]$ -module \tilde{M} in Lemma 5.2.

The proof will be given in (5.10), at the end of this section.

Corollary 5.4. *Assuming the truth of Theorem 5.3, both the split and the quasi-split version of Kottwitz' Conjecture 2.8 hold for W_n .*

Proof. As far as the split version is concerned, the argument is given in [1, Cor. 7.6]; this also uses the formulae in (3.9) for $\Upsilon_{\sigma_{n/2}}^1$ and $\Upsilon_{t\sigma_{n/2}t}^1$ (where the signs ν_α have been fixed). Now consider the quasi-split case. Let C be a \diamond -conjugacy class of \diamond -twisted involutions of W_n . Then $\mathcal{C} := \{tw \mid w \in C\}$ is a conjugacy class of involutions of \tilde{W}_n which is contained in the coset tW_n ; see Remark 2.4. Let Γ be a left cell of W_n and $\tilde{\Gamma} = \Gamma \cup t\Gamma$

be the corresponding left L -cell of \tilde{W}_n . Then, using (4.2)(b), Lemma 5.2(c), Frobenius reciprocity and Theorem 5.3, we obtain

$$\langle \Upsilon_C^\circ, [\Gamma]_1 \rangle_{W_n} = \langle \tilde{\Upsilon}_C, [\tilde{\Gamma}]_1 \rangle_{\tilde{W}_n} = |\mathcal{C} \cap \tilde{\Gamma}|.$$

Since $\mathcal{C} \subseteq tW_n$, the right hand side equals $|\mathcal{C} \cap t\Gamma| = |t(C \cap \Gamma)| = |C \cap \Gamma|$, as required. \square

We now turn to the proof of Theorem 5.3; this will require a number of preparations.

Proposition 5.5. *Let \mathcal{C} be a conjugacy class of involutions in \tilde{W}_n . Assume that $\sigma_{l,j} \in \mathcal{C}$ where $l, j \geq 0$ are such that $l + 2j \leq n$; see (3.10). Then, using the notation in (3.11), we have*

$$\tilde{\Upsilon}_C = \sum_{(\alpha, \beta)} \binom{c(\alpha, \beta)}{j + l - d_0(\alpha, \beta)} \tilde{\chi}^{(\alpha, \beta)}$$

where the sum runs over all $(\alpha, \beta) \vdash n$ such that $\tilde{\chi}^{(\alpha, \beta)}$ is \diamond -special, $|\beta| = j$ and $d_0(\alpha, \beta) \leq j + l \leq d_0(\alpha, \beta) + c(\alpha, \beta)$.

Proof. Assume first that n is even, $l = 0$ and $j = n/2$. Then, by (3.9) and Lemma 5.2(b), the restriction of $\tilde{\Upsilon}_C$ to W_n equals

$$\Upsilon_{\sigma_{n/2}}^1 + \Upsilon_{t\sigma_{n/2}t}^1 = \sum_{\alpha \vdash n/2} (\chi^{[\alpha, +]} + \chi^{[\alpha, -]}).$$

So Frobenius reciprocity immediately implies that $\tilde{\Upsilon}_C = \sum_{\alpha \vdash n/2} \chi^{(\alpha, \alpha)}$, in accordance with the formula stated above. Now assume that $j < n/2$ and let (α, β) be a pair of partitions such that $|\alpha| + |\beta| = n$. Then Frobenius reciprocity, Proposition 3.12 and Lemma 5.2 show that

$$\langle \tilde{\Upsilon}_C, \text{Ind}_{W_n}^{\tilde{W}_n}(\chi^{[\alpha, \beta]}) \rangle_{\tilde{W}_n} = 0 \quad \text{unless} \quad \alpha \neq \beta, |\beta| = j \text{ and } \tilde{\chi}^{(\alpha, \beta)} \text{ is } \diamond\text{-special};$$

furthermore, if $\alpha \neq \beta$, $|\beta| = j$ and $\tilde{\chi}^{(\alpha, \beta)}$ is \diamond -special, then

$$\langle \tilde{\Upsilon}_C, \text{Ind}_{W_n}^{\tilde{W}_n}(\chi^{[\alpha, \beta]}) \rangle_{\tilde{W}_n} = \text{binomial coefficient as above.}$$

Now let $\alpha \neq \beta$. Since

$$\text{Ind}_{W_n}^{\tilde{W}_n}(\chi^{[\alpha, \beta]}) = \tilde{\chi}^{(\alpha, \beta)} + \tilde{\chi}^{(\beta, \alpha)},$$

it will be sufficient to show that

$$(*) \quad \langle \tilde{\Upsilon}_C, \tilde{\chi}^{(\alpha, \beta)} \rangle_{\tilde{W}_n} = 0 \quad \text{unless} \quad |\alpha| > |\beta|.$$

Now, by Example 3.10, we can assume that $\sigma = \sigma_{l,j}$ is the longest element in a parabolic subgroup $\tilde{W}_I \subseteq \tilde{W}_n$ where $I \subseteq \{t, s_1, \dots, s_{n-1}\}$; furthermore, σ is central in \tilde{W}_I . Then $C_{\tilde{W}_n}(\sigma) = \tilde{Y} \rtimes \tilde{W}_I$ where \tilde{Y} is a certain set of distinguished coset representatives of \tilde{W}_I in \tilde{W}_n ; see [6, §2]. By the argument in [6, Lemma 2.1] (see also [6, Rem. 3.3]; this is essentially the same argument as in (2.6)), we have

$$\tilde{M} \cong \text{Ind}_{C_{\tilde{W}_n}(\sigma)}^{\tilde{W}_n}(\tilde{\varepsilon}_\sigma)$$

where the homomorphism $\tilde{\varepsilon}_\sigma: C_{\tilde{W}_n}(\sigma) \rightarrow \{\pm 1\}$ is given by

$$\tilde{\varepsilon}_\sigma(yw') = (-1)^{\tilde{\ell}(w') - \ell_t(w')} \quad \text{for all } y \in \tilde{Y} \text{ and } w' \in \tilde{W}_I.$$

Then $(*)$ is shown in [6, Lemma 3.4]. (Note that, in [6], it is generally assumed that $\mathcal{C} \subseteq W_n$ but in the proof of [6, Lemma 3.4], this assumption is irrelevant.) \square

Remark 5.6. Let $X \subseteq \tilde{W}_n$ be any subset which is a union of (ordinary) conjugacy classes of involutions in \tilde{W}_n . Then we set

$$\tilde{\Upsilon}_X = \sum_{\mathcal{C}} \tilde{\Upsilon}_{\mathcal{C}}$$

where \mathcal{C} runs over the conjugacy classes contained in X . In particular, this applies to the set of all involutions in \tilde{W}_n , which we denote by \mathcal{I}_n . With this notation, we have

$$\tilde{\Upsilon}_{\mathcal{I}_n} = \sum_{\tilde{\chi} \in \text{Irr}(\tilde{W}_n) \text{ } \diamond\text{-special}} 2^{c(\tilde{\chi})} \tilde{\chi}.$$

This immediately follows from Proposition 5.5, by summing over all $l, j \geq 0$ such that $l + 2j \leq n$.

Lemma 5.7. *Let $\tilde{\mathfrak{C}}$ be a two-sided L -cell of \tilde{W}_n and \mathcal{C} be any ordinary conjugacy class of involutions in \tilde{W}_n . Then*

$$|\mathcal{C} \cap \tilde{\mathfrak{C}}| = \tilde{\chi}_0(1) |\mathcal{C} \cap \tilde{\Gamma}| \quad \text{for any left } L\text{-cell } \tilde{\Gamma} \subseteq \tilde{\mathfrak{C}},$$

where $\tilde{\chi}_0 \in \text{Irr}(\tilde{W}_n)$ is the unique \diamond -special character in $\text{Irr}(\tilde{W}_n \mid \tilde{\mathfrak{C}})$.

Proof. Let \mathfrak{C} be a two-sided cell of W_n such that $\tilde{\mathfrak{C}} = \mathfrak{C}^+$; see (4.2). Then we also have $\tilde{\Gamma} = \Gamma \cup t\Gamma$ for some left cell $\Gamma \subseteq \mathfrak{C}$ of W_n . Let $\chi_0 \in \text{Irr}(W_n \mid \mathfrak{C})$ be the unique special character. By [1, Exp. 4.5], we already know that

$$(*) \quad |\mathcal{C} \cap \mathfrak{C}| = \chi_0(1) |\mathcal{C} \cap \Gamma| \quad \text{if } \mathcal{C} \subseteq W_n.$$

We now distinguish two cases.

Case 1. Assume that $\mathfrak{C}^\diamond = \mathfrak{C}$. Then $\tilde{\mathfrak{C}} = \mathfrak{C}^+ = \mathfrak{C} \cup t\mathfrak{C}$. Furthermore, $\tilde{\chi}_0$ is the preferred extension of χ_0 ; in particular, $\tilde{\chi}_0(1) = \chi_0(1)$. Now, if $\mathcal{C} \subseteq W_n$, then $\mathcal{C} \cap \tilde{\mathfrak{C}} = \mathcal{C} \cap \mathfrak{C}$ and $\mathcal{C} \cap \tilde{\Gamma} = \mathcal{C} \cap \Gamma$. So the assertion holds by $(*)$.

On the other hand, if $\mathcal{C} \subseteq tW_n$, then $\mathcal{C} := \{t\sigma \mid \sigma \in \mathcal{C}\}$ is a \diamond -conjugacy class of \diamond -twisted involutions in W_n . In this case, we have $\mathcal{C} \cap \tilde{\mathfrak{C}} = t(\mathcal{C} \cap \mathfrak{C})$ and $\mathcal{C} \cap \tilde{\Gamma} = t(\mathcal{C} \cap \Gamma)$. So the assertion holds by Proposition 4.11(b).

Case 2. Assume that $\mathfrak{C}^\diamond \neq \mathfrak{C}$. Then $\tilde{\mathfrak{C}} = \mathfrak{C}^+ = \mathfrak{C} \cup t\mathfrak{C} \cup \mathfrak{C}t \cup t\mathfrak{C}t$. If $\mathcal{C} \subseteq tW_n$, then both sides of the desired identity are zero. (Indeed, we have $\mathfrak{C}^{-1} = \mathfrak{C}$ by [13, 5.2(iii)] and so, if $1 = (tw)^2 = twtw = w^\diamond w$, then $w^\diamond = w^{-1} \in \mathfrak{C} \cap \mathfrak{C}^\diamond$, a contradiction.) Now assume that $\mathcal{C} \subseteq W_n$. Then

$$|\mathcal{C} \cap \tilde{\mathfrak{C}}| = |\mathcal{C} \cap \mathfrak{C}| + |\mathcal{C} \cap t\mathfrak{C}t| = 2 |\mathcal{C} \cap \mathfrak{C}|.$$

By $(*)$, we have $|\mathcal{C} \cap \mathfrak{C}| = \chi_0(1) |\mathcal{C} \cap \Gamma|$ and so $|\mathcal{C} \cap \tilde{\mathfrak{C}}| = \chi_0(1) |\mathcal{C} \cap \tilde{\Gamma}|$. It remains to note that, since $\mathfrak{C}^\diamond \neq \mathfrak{C}$, we have $\chi_0^\diamond \neq \chi_0$ and so $\tilde{\chi}_0$ is obtained by inducing χ_0 from W_n to \tilde{W}_n ; in particular, $\tilde{\chi}_0(1) = 2\chi_0(1)$. \square

Let $\tilde{\mathfrak{C}}$ be a two-sided L -cell of \tilde{W}_n . We say that “*Kottwitz’ Modified Conjecture holds for $\tilde{\mathfrak{C}}$* ” if, for any conjugacy class of involutions \mathcal{C} in \tilde{W}_n , we have

$$\langle \tilde{\Upsilon}_{\mathcal{C}}, [\tilde{\Gamma}]_1 \rangle_{\tilde{W}_n} = |\mathcal{C} \cap \tilde{\Gamma}| \quad \text{for all left } L\text{-cells } \tilde{\Gamma} \subseteq \tilde{\mathfrak{C}}.$$

The following remark, together with (5.9), will provide the basis for an inductive proof of Theorem 5.3 where we proceed one two-sided L -cell at a time.

5.8. Let $\tilde{w}_0 \in \tilde{W}_n$ be the longest element. Let \mathcal{C} be a conjugacy class of involutions of \tilde{W}_n . Let \tilde{M} be the corresponding $\mathbb{Q}[\tilde{W}_n]$ -module as in Lemma 5.2. Since \tilde{w}_0 is central in \tilde{W}_n , the set $\mathcal{C}\tilde{w}_0$ also is a conjugacy class of involutions in \tilde{W}_n . Let \tilde{M}_0 be the corresponding $\mathbb{Q}[\tilde{W}_n]$ -module. Then we have

$$(a) \quad \tilde{M}_0 \cong \tilde{M} \otimes \varepsilon' \quad \text{and} \quad \tilde{\Upsilon}_{\mathcal{C}\tilde{w}_0} = \tilde{\Upsilon}_{\mathcal{C}} \otimes \varepsilon',$$

where $\varepsilon': W_n \rightarrow \{\pm 1\}$ is the homomorphism given by $\varepsilon'(t) = 1$ and $\varepsilon'(s_i) = -1$ for $1 \leq i \leq n-1$. This follows by an argument entirely analogous to that in [1, Lemma 5.2]. Now let $\tilde{\Gamma}$ be a left L -cell of \tilde{W}_n . Then $\tilde{\Gamma}\tilde{w}_0$ also is a left L -cell of \tilde{W}_n ; see [17, 11.7]. We show that

$$(b) \quad [\tilde{\Gamma}\tilde{w}_0]_1 = [\tilde{\Gamma}]_1 \otimes \varepsilon'.$$

Indeed, by (4.2)(b), $[\tilde{\Gamma}]_1$ is obtained by inducing $[\Gamma]_1$ from W_n to \tilde{W}_n where Γ is a left cell of W_n such that $\tilde{\Gamma} = \Gamma \cup t\Gamma$. If n is even, then $\tilde{w}_0 \in W_n$ and this is the longest element in W_n . So $\tilde{\Gamma}\tilde{w}_0 = (\Gamma\tilde{w}_0) \cup t(\Gamma\tilde{w}_0)$. By [13, Lemma 5.14], we have $[\Gamma\tilde{w}_0]_1 = [\Gamma]_1 \otimes \varepsilon$ where ε is the sign character of W_n . Since ε is the restriction of ε' to W_n , this implies (b) in this case. Now assume that n is odd. Then $w_0 := t\tilde{w}_0 \in W_n$ and this is the longest element in W_n ; furthermore, $\Gamma = \Gamma^\circ = t\Gamma t$. So we obtain $\tilde{\Gamma}\tilde{w}_0 = \Gamma w_0 \cup t(\Gamma w_0)$. Using once more [13, Lemma 5.14], this implies (b) in this case as well. We conclude that

$$(c) \quad \langle \tilde{\Upsilon}_{\mathcal{C}}, [\tilde{\Gamma}]_1 \rangle_{\tilde{W}_n} = \langle \tilde{\Upsilon}_{\mathcal{C}\tilde{w}_0}, [\tilde{\Gamma}\tilde{w}_0]_1 \rangle_{\tilde{W}_n} \quad \text{for any left } L\text{-cell } \tilde{\Gamma} \text{ of } \tilde{W}_n.$$

In particular, Kottwitz' Modified Conjecture holds for a two-sided L -cell $\tilde{\mathfrak{C}}$ if and only if it holds for the two-sided L -cell $\tilde{\mathfrak{C}}\tilde{w}_0$.

Next, there is a standard combinatorial procedure by which certain arguments about two-sided cells can be reduced to so-called “cuspidal” two-sided cells. This appears and is used at various places in Lusztig's work; see, for example, [13, 8.1] and [15, 17.13]. We have also used it in [1, §6] to deal with Kottwitz' conjecture in type B_n . Let us explicitly describe this procedure in our present context.

5.9. For any $r \in \{0, 1, \dots, n\}$, we have a parabolic subgroup $\tilde{W}_{n-r} = \langle t, s_1, \dots, s_{n-r-1} \rangle \subseteq \tilde{W}_n$ of type B_{n-r} (where $\tilde{W}_0 = \{1\}$ and $\tilde{W}_1 = \{1, t\}$). Following Lusztig [12, 1.8], [13, 4.1], we define an additive map

$$\mathbf{J}_r: \mathbb{Z}[\text{Irr}(\tilde{W}_{n-r})] \rightarrow \mathbb{Z}[\text{Irr}(\tilde{W}_n)]$$

as follows. Consider the parabolic subgroup $\tilde{W}' = \tilde{W}_{n-r} \times H_r \subseteq \tilde{W}_n$ where $H_r = \langle s_{n-r+1}, \dots, s_{n-1} \rangle \cong \mathfrak{S}_r$. Let $\tilde{\chi}' \in \text{Irr}(\tilde{W}_{n-r})$ and ε_r be the sign character on H_r . Since $\mathbf{a}_{\varepsilon_r} = r(r-1)/2$, we have the implication

$$\left\langle \text{Ind}_{\tilde{W}'}^{\tilde{W}_n}(\tilde{\chi}' \boxtimes \varepsilon_r), \tilde{\chi} \right\rangle_{\tilde{W}_n} \neq 0 \quad \Rightarrow \quad \mathbf{a}_{\tilde{\chi}} \geq \mathbf{a}_{\tilde{\chi}'} + r(r-1)/2$$

for all $\tilde{\chi} \in \text{Irr}(\tilde{W}_n)$. We set

$$\mathbf{J}_r(\tilde{\chi}') := \sum_{\tilde{\chi}} \left\langle \text{Ind}_{\tilde{W}'}^{\tilde{W}_n}(\tilde{\chi}' \boxtimes \varepsilon_r), \tilde{\chi} \right\rangle_{\tilde{W}_n} \tilde{\chi}$$

where the sum runs over all $\tilde{\chi} \in \text{Irr}(\tilde{W}_n)$ such that $\mathbf{a}_{\tilde{\chi}} = \mathbf{a}_{\tilde{\chi}'} + r(r-1)/2$. Now let $\tilde{\mathfrak{C}}$ be a two-sided L -cell of \tilde{W}_n . Following [13, 8.1], [15, 17.13], we say that $\tilde{\mathfrak{C}}$ is “*smoothly induced*” if there exists some $r \in \{1, \dots, n\}$ and a two-sided cell L -cell $\tilde{\mathfrak{C}}'$ of \tilde{W}_{n-r} such that \mathbf{J}_r establishes a bijection

$$(a) \quad \text{Irr}(\tilde{W}_{n-r} \mid \tilde{\mathfrak{C}}') \rightarrow \text{Irr}(\tilde{W}_n \mid \tilde{\mathfrak{C}}), \quad \tilde{\chi}' \mapsto \mathbf{J}_r(\tilde{\chi}').$$

As in [1, Remark 6.2], one easily sees that then the following holds:

$$(b) \quad c(\tilde{\chi}_0) = c(\tilde{\chi}'_0), \quad d_0(\tilde{\chi}_0) = d_0(\tilde{\chi}'_0) + \lfloor r/2 \rfloor \quad \text{and} \quad |\beta| = |\beta'| + \lfloor r/2 \rfloor,$$

where $\tilde{\chi}_0 = \tilde{\chi}^{(\alpha, \beta)} \in \text{Irr}(\tilde{W}_n \mid \tilde{\mathfrak{C}})$ and $\tilde{\chi}'_0 = \tilde{\chi}^{(\alpha', \beta')} \in \text{Irr}(\tilde{W}_{n-r} \mid \tilde{\mathfrak{C}}')$ are the unique \diamond -special characters. Now $\{w_{0,r}\}$ is a two-sided cell in H_r where $w_{0,r} \in H_r$ is the longest element; furthermore, $\text{Irr}(H_r \mid \{w_{0,r}\}) = \{\varepsilon_r\}$. Consequently, $\tilde{\mathfrak{C}}'w_{0,r} \subseteq \tilde{W}'$ is a two-sided L -cell (with respect to the restriction of L to \tilde{W}'); thus, using also [18, 43.11(b)], we have

$$(c) \quad \tilde{\mathfrak{C}}'w_{0,r} \subseteq \tilde{\mathfrak{C}} \quad \text{and} \quad \text{Irr}(\tilde{W}' \mid \tilde{\mathfrak{C}}'w_{0,r}) = \{\tilde{\chi}' \boxtimes \varepsilon_r \mid \tilde{\chi}' \in \text{Irr}(\tilde{W}_{n-r} \mid \tilde{\mathfrak{C}}')\}.$$

Finally, the point of these definitions is that every two-sided L -cell $\tilde{\mathfrak{C}}$ of \tilde{W}_n is either itself smoothly induced, or the two-sided L -cell $\tilde{\mathfrak{C}}\tilde{w}_0$ is smoothly induced (where $\tilde{w}_0 \in \tilde{W}_n$ is the longest element), or $n = d^2$ for some $d \geq 2$ in which case $\tilde{\mathfrak{C}}$ is uniquely determined; in the last case, $\tilde{\mathfrak{C}}$ is called “*cuspidal*” and determined by the condition that the unique \diamond -special character in $\text{Irr}(\tilde{W}_n \mid \tilde{\mathfrak{C}})$ is $\tilde{\chi}^{(\alpha, \beta)}$ where $\alpha = (1, 2, \dots, d-1)$ and $\beta = (0, 1, 2, \dots, d-2)$; see [12, 3.17], [13, 8.1].

5.10. Proof of Theorem 5.3. We proceed by induction on n . If $n = 0$, then $\tilde{W}_0 = \{1\}$ and the assertion is clear. Now assume that $n \geq 1$. Let $\tilde{\mathfrak{C}}$ be a two-sided L -cell of \tilde{W}_n . Assume first that $\tilde{\mathfrak{C}}$ is smoothly induced; let $r \in \{1, \dots, n\}$ and $\tilde{\mathfrak{C}}' \subseteq \tilde{W}_{n-r}$ be as in (5.9). Let \mathcal{C} be any conjugacy class of involutions in \tilde{W}_n . Assume that $\sigma_{l,j} \in \mathcal{C}$ where $l, j \geq 0$ are such that $l + 2j \leq n$; we write $\mathcal{C} = \mathcal{C}_{l,j}$. Let $\tilde{\chi}_0 = \tilde{\chi}^{(\alpha, \beta)} \in \text{Irr}(\tilde{W}_n \mid \tilde{\mathfrak{C}})$ be the unique \diamond -special character. Let $\tilde{\Gamma}$ be a left L -cell of \tilde{W}_n such that $\tilde{\Gamma} \subseteq \tilde{\mathfrak{C}}$. By Example 4.6(d), we have $\langle [\tilde{\Gamma}]_1, \tilde{\chi}_0 \rangle_{\tilde{W}_n} = 1$. Hence, using Proposition 5.5, we already know that

$$\langle \tilde{\Upsilon}_{\mathcal{C}_{l,j}}, [\tilde{\Gamma}]_1 \rangle_{\tilde{W}_n} = \begin{cases} \begin{pmatrix} c(\alpha, \beta) \\ j + l - d_0(\alpha, \beta) \end{pmatrix} & \text{if } |\beta| = j, \\ 0 & \text{otherwise.} \end{cases}$$

We will now show that

$$(\dagger) \quad |\mathcal{C}_{l,j} \cap \tilde{\Gamma}| \leq \langle \tilde{\Upsilon}_{\mathcal{C}_{l,j}}, [\tilde{\Gamma}]_1 \rangle_{\tilde{W}_n}.$$

This is seen as follows. If $\mathcal{C}_{l,j} \cap \tilde{\Gamma} = \emptyset$, this is clear. Now assume that $\mathcal{C}_{l,j} \cap \tilde{\Gamma} \neq \emptyset$. By Lemma 5.7, the cardinality $|\mathcal{C}_{l,j} \cap \tilde{\Gamma}|$ does not depend on the left L -cell $\tilde{\Gamma} \subseteq \tilde{\mathfrak{C}}$. Thus, it will be enough to prove (\dagger) for one particular left L -cell in $\tilde{\mathfrak{C}}$. We will choose such a left L -cell as follows. Consider the two-sided L -cell $\tilde{\mathfrak{C}}'$ of \tilde{W}_{n-r} . As in (5.9), let $w_{0,r} \in H_r$ be the longest element. Then $\tilde{\mathfrak{C}}'w_{0,r}$ is a two-sided L -cell in $\tilde{W}' = \tilde{W}_{n-r} \times H_r$ and we have $\tilde{\mathfrak{C}}'w_{0,r} \subseteq \tilde{\mathfrak{C}}$; see (5.9)(c). Now let $\tilde{\Gamma}'$ be a left L -cell of \tilde{W}_{n-r} which is contained in $\tilde{\mathfrak{C}}$. Then $\tilde{\Gamma}'w_{0,r}$ is a left L -cell of \tilde{W}' . Let $\tilde{\Gamma} \subseteq \tilde{\mathfrak{C}}$ be the left L -cell of \tilde{W}_n such that

$$\tilde{\Gamma}'w_{0,r} \subseteq \tilde{\Gamma}.$$

By Example 4.6(d) and (5.9)(b), the left L -cells $\tilde{\Gamma}'$, $\tilde{\Gamma}'w_{0,r}$ and $\tilde{\Gamma}$ all contain the same number of involutions. Hence, all the involutions in $\tilde{\Gamma}$ are already contained in $\tilde{\Gamma}'w_{0,r} \subseteq \tilde{W}'$. Consequently, we have

$$\mathcal{C}_{l,j} \cap \tilde{\Gamma} = (\mathcal{C}_{l,j} \cap \tilde{W}') \cap \tilde{\Gamma}'w_{0,r}.$$

By an argument entirely analogous to that in [1, Theorem 6.3] (see the paragraph following (Δ) in the proof thereof), the assumption that $\mathcal{C}_{l,j} \cap \tilde{\Gamma} \neq \emptyset$ now implies that $\mathcal{C}_{l,j} \cap \tilde{W}'$ is the conjugacy class containing the element $\sigma_{l,j'}w_{0,r}$ where $j = j' + k$ and $k = \lfloor r/2 \rfloor$. We conclude that

$$|\mathcal{C}_{l,j} \cap \tilde{\Gamma}| = |\mathcal{C}' \cap \tilde{\Gamma}'| \quad \text{where } \mathcal{C}' \subseteq \tilde{W}_{n-r} \text{ is the conjugacy class containing } \sigma_{l,j'}.$$

Hence, using the equality $j' = j - \lfloor r/2 \rfloor$ and (5.9)(b), we see that (\dagger) holds by induction. Once this is established, it actually follows that we must have equality in (\dagger) . Indeed, by Example 4.6(d), we have

$$\sum_{l,j} |\mathcal{C}_{l,j} \cap \tilde{\Gamma}| = (\text{number of involutions in } \tilde{\Gamma}) = 2^{c(\tilde{\chi}_0)}$$

where the sum runs over all $l, j \geq 0$ such that $l + 2j \leq n$. But we obtain the same result when we sum the binomial coefficients giving the right hand side of (\dagger) over all l, j as above. Hence, all the inequalities in (\dagger) must be equalities. Thus, Kottwitz' Modified Conjecture holds for $\tilde{\mathfrak{C}}$. Then (5.8) shows that Kottwitz' Modified Conjecture also holds for $\tilde{\mathfrak{C}}\tilde{w}_0$. By the remarks at the end of (5.9), these arguments cover all non-cuspidal two-sided L -cells of \tilde{W}_n . So it remains to show that Kottwitz' Modified Conjecture holds for the unique cuspidal two-sided L -cell of \tilde{W}_n where $n = d^2$ for some $d \geq 2$. But this follows from a formal argument based on Lemma 5.7, exactly as in the proof of [1, Theorem 6.3]. \square

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